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## ► To cite this version:

Xavier Goaoc, Hyo-Sil Kim, Sylvain Lazard. Bounded-Curvature Shortest Paths through a Sequence of Points. [Research Report] RR-7465, INRIA. 2010, pp.53. inria-00539957

**HAL Id: inria-00539957**

**<https://inria.hal.science/inria-00539957>**

Submitted on 29 Nov 2010

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# ***Bounded-Curvature Shortest Paths through a Sequence of Points***

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**N° 7465**

November 2010

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 ***rapport  
de recherche***



# Bounded-Curvature Shortest Paths through a Sequence of Points

Xavier Goaoc<sup>\*</sup>, Hyo-Sil Kim<sup>†</sup>, Sylvain Lazard<sup>\*</sup>

Thème : Algorithmique, calcul certifié et cryptographie  
Équipe-Projet Végas

Rapport de recherche n° 7465 — November 2010 — 50 pages

**Abstract:** We consider the problem of computing shortest paths having curvature at most one almost everywhere and visiting a sequence of  $n$  points in the plane in a given order. This problem arises naturally in path planning for point car-like robots in the presence of polygonal obstacles, and is also a sub-problem of the Dubins Traveling Salesman Problem.

This problem reduces to minimizing the function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  that maps  $(\theta_1, \dots, \theta_n)$  to the length of a shortest curvature-constrained path that visits the points  $p_1, \dots, p_n$  in order and whose tangent in  $p_i$  makes an angle  $\theta_i$  with the  $x$ -axis. We show that when consecutive points are distance at least 4 apart, all minima of  $F$  are realized over at most  $2^k$  disjoint convex polyhedra over which  $F$  is strictly convex; each polyhedron is defined by  $4n - 1$  linear inequalities and  $k$  denotes, informally, the number of  $p_i$  such that the angle  $\angle(p_{i-1}, p_i, p_{i+1})$  is small. A curvature-constrained shortest path visiting a sequence points can therefore be approximated by standard convex optimization methods, which presents an interesting alternative to the known polynomial-time algorithms that can only compute a multiplicative constant factor approximation.

Our technique also opens new perspectives for bounded-curvature path planning among polygonal obstacles. In particular, we show that, under certain conditions, if the sequence of points where a shortest path touches the obstacles is known then “connecting the dots” reduces to a family of convex optimization problems.

**Key-words:** Path planning, Bounded curvature, Dubins path, Convex optimization.

This work was supported by the INRIA *Équipe Associée* KI, the BK21 project, and Mid-career Researcher Program through NRF grant funded by the MEST (No. R01-2008-000-11607-0).

<sup>\*</sup> INRIA Nancy Grand Est, LORIA laboratory, Nancy, France. `firstname.name@loria.fr`

<sup>†</sup> Department of Computer Science, KAIST, Daejeon, Korea. `hyosil@tclab.kaist.ac.kr`

# Plus Court Chemin de Courbure Bornée Passant par une Séquence de Points

**Résumé :** Nous considérons le problème du calcul d'un plus court chemin de courbure au plus 1 presque partout et visitant une séquence de points dans un ordre donné. Cette question apparaît naturellement en planification de trajectoire pour des véhicules de type voiture en présence d'obstacles polygonaux ainsi que dans le *problème du voyageur de commerce de Dubins*.

Ce problème se ramène à minimiser une fonction  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  qui à  $(\theta_1, \dots, \theta_n)$  associe la longueur du plus court chemin de courbure bornée qui visite les points  $p_1, \dots, p_n$  dans cet ordre et dont la tangente en  $p_i$  fait un angle  $\theta_i$  avec l'axe des abscisses. Nous montrons que quand la distance entre points consécutifs est au moins 4, tous les minima de  $F$  sont atteints sur un domaine qui est l'union disjointe d'au plus  $2^k$  polyèdres sur lesquels  $F$  est strictement convexe; chaque polyèdre est défini par  $4n - 1$  inégalités linéaires et  $k$  représente, informellement, le nombre de points  $p_i$  tels que l'angle  $\angle(p_{i-1}, p_i, p_{i+1})$  est faible. On peut ainsi approximer un plus court chemin de courbure borné visitant une séquence de points par des techniques standard d'optimisation convexe. Cela représente une alternative intéressante aux méthodes antérieures, qui ne peuvent garantir qu'une approximation à un facteur constant multiplicatif.

Notre technique ouvre aussi de nouvelles perspectives concernant la planification de trajectoire à courbure bornée en présence d'obstacles polygonaux. En particulier, nous montrons que sous certaines conditions, si la séquence des points en lesquels le chemin touche les obstacles est connue alors relier ces points se ramène à une famille de problèmes d'optimisation convexe.

**Mots-clés :** Planification de trajectoire, Courbure bornée, Chemins de Dubins, Optimisation convexe

## 1 Introduction

*Path-planning* problems involve computing feasible paths, possibly optimal for some criterion such as time or length, for a robot moving among obstacles. These problems are central in robotics and they have been widely studied; see, for instance, the books and survey papers [15, 18, 19, 29]. In its simplest form, path planning focuses on collision-free paths. However, robots generally come with physical limitations, such as bounds on the velocity, acceleration or curvature. Such differential constraints, called *nonholonomic*, restrict the geometry of the paths it can follow. Although there has been a considerable amount of work on nonholonomic motion planning in the robotics and control communities, relatively little work has been done, in comparison, from an algorithmic perspective.

In this paper, we study the path-planning problem for a car-like robot. The robot *configuration* is specified by both its location, a point  $p$  in  $\mathbb{R}^2$  (typically, the midpoint of the rear axle), and its direction of travel which we represent by its polar angle  $\theta$  in  $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ . The robot is constrained to move in the forward direction, and its turning radius is bounded from below by a positive constant, which can be assumed to be equal to one by scaling the space. In this context, the robot follows *bounded-curvature* paths, that is, differentiable curves whose curvature is constrained to be at most one almost everywhere. Furthermore, for any robot configuration  $(p, \theta)$ , the oriented tangent to the curve at  $p$  has a polar angle  $\theta$ .

The first results on curvature-constrained shortest paths go back to Dubins [12] in 1957. He proved that, in the plane without obstacles, bounded-curvature shortest paths consist of arcs of unit radius circles (*C*-segments) and straight line segments (*S*-segments); moreover, such shortest paths are of type *CCC* or *CSC*, or a substring thereof. These types of paths are generally referred to as *Dubins paths*.

We consider the problem of computing bounded-curvature shortest paths that visit, in order, a given sequence of  $n$  points in the plane (with no obstacles). This question is related to the problem of path planning in the presence of polygonal obstacles, because, roughly speaking, such a shortest path is also a locally shortest path through a sequence of points in the absence of obstacles (see below for details). This problem is also a sub-problem of the Dubins Traveling Salesman Problem which has been substantially studied in the robotics literature, for instance, in the context of UAV (unmanned air vehicles) path planning.

**Our results.** Let  $p_1, \dots, p_n$  be a sequence of points in the plane. For  $1 < i < n$ , we say that  $p_i$  is a *sharp turn* if the angle  $\angle(p_{i-1}, p_i, p_{i+1})$  is acute, and one of  $p_i$ 's neighbors,  $p_{i-1}$  or  $p_{i+1}$ , is within distance 4 from the segment joining  $p_i$  to its other neighbor. Let  $F : (\mathbb{S}^1)^n \rightarrow \mathbb{R}$  map a sequence  $(\theta_1, \dots, \theta_n)$  of angles to the length of a shortest curvature-constrained path visiting the configurations  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  in order. Since  $\mathbb{S}^1$  can be *lifted to* (that is, represented by) some interval of length  $2\pi$  in  $\mathbb{R}$ , we can also see  $F$  as a function from (a subset of)  $\mathbb{R}^n$  to  $\mathbb{R}$ . Dubins' characterization implies that a curvature-constrained shortest path between two configurations can be computed in constant time. Computing a curvature-constrained shortest path visiting the points in order is thus equivalent, from a computational point of view, to finding a minimum of the function  $F$ . Our main result is the following:

**Theorem 1.** *Let  $p_1, \dots, p_n$  be a sequence of points in the plane that has  $k$  sharp turns and such that any two consecutive points are at least distance 4 apart. All global minima of  $F$  are realized in an open domain of  $(\mathbb{S}^1)^n$  that can be lifted to a union of up to  $2^k$  disjoint convex polyhedra in  $\mathbb{R}^n$ , each defined by at most  $4n - 1$  linear inequalities, in two variables each.<sup>1</sup> Moreover, through this lifting,  $F$  is strictly convex over each of these polyhedra.*

Theorem 1 implies that a curvature-constrained shortest path visiting the points in order can be approximated by standard methods for convex optimization problems. Since convex optimization methods are known to be efficient in practice, this appears to be an interesting alternative to the known polynomial-time algorithms, which can only compute a *multiplicative* constant factor approximation. It should also be stressed that Theorem 1 easily extends to the case where the initial and final directions  $\theta_1$  and  $\theta_n$  are prescribed (Theorem 28).

The technique we develop to prove Theorem 1 opens the way to a new approach to path planning among polygonal obstacles. It is known that a bounded-curvature shortest path between *two* configurations in the presence of polygonal obstacles is a concatenation of Dubins paths whose extremities are

<sup>1</sup>Note that, through the lifting,  $F$  remains continuous and strictly convex on the closure of each polyhedron.

extremal configurations or contact points on the boundary of the obstacles [14, 17]. We show that, given the sequence of contact points and knowing which one lie on *anchored* circular arcs (i.e., arcs of the path that touch the obstacle more than once), the reconstruction of the whole path reduces, under certain conditions, to a family of convex optimization problems (Theorem 31); in other words, “connecting the dots” is now easy, and the difficult task appears to be the discrete subproblem of computing the contact points and the anchored circular arcs.

We also establish several new properties of shortest paths of bounded curvature. In particular, we prove two fundamental results on Dubins paths of type *CSC*. First, if a *CSC*-path joining two configurations is such that its two circular arcs are shorter than  $\pi$ , then it is necessarily the shortest of all the *CSC*-paths (Proposition 5). Also, given any two points  $p_1$  and  $p_2$ , the length  $F(\theta_1, \theta_2)$  of the shortest *CSC*-path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is  $C^2$  and locally strictly convex over the domain where the two circular arcs are shorter than  $\pi$  (Theorem 6). We also prove that the length function  $F$  may have  $2^{n-2}$  local minima (Corollary 10), each of them corresponding to a distinct *locally shortest path* in which the length of the circular arcs preceding and following every point  $p_i$ ,  $1 < i < n$ , are equal or sum up to  $2\pi$  (Proposition 9).

Our results say little on the theoretical complexity of approximating the shortest curvature-constrained path through a sequence of points since most convex optimization methods are notoriously hard to analyze. Nevertheless, in the absence of sharp turn ( $k = 0$ ), and if consecutive points in the input are distance at least  $\varsigma$  apart for some  $\varsigma > 2 + \sqrt{5} \sim 4.24$ , the ellipsoid method, although known to be inefficient both in theory and in practice, can be analyzed: it computes in  $O(n^4 \log \frac{n}{\varepsilon})$  time (in an extended real RAM model) a curvature-constrained path that visits  $p_1, \dots, p_n$  in order and whose length exceeds that of the (unique) optimum by at most  $\varepsilon$  (Corollary 24).

**Previous work.** As mentioned above, the study of bounded-curvature path planning started with Dubins’ [12] first characterization of the geometry of shortest paths of bounded curvature, in the plane without obstacles. A more direct proof of this result, using ideas from control theory, was presented later by Boissonnat et al. [6] and Sussmann and Tang [32], independently.

The problem of computing shortest paths of bounded curvature through an ordered sequence of points was, to our knowledge, first considered by Bui [9] in 1994. Bui’s high-level approach was similar to ours, that is to argue that a shortest path corresponds to a minimum of a convex function; unfortunately, several of the proofs from [9] have serious gaps. Bui also showed that a path of minimal length corresponds to a solution of one of  $2^n$  algebraic systems, each consisting of  $O(n)$  equations of bounded degree; even though this is totally unpractical, this solution illustrates well the difficulty of the problem.

An approximate solution can easily be obtained by considering the points as given on-line and greedily concatenating shortest paths from configuration  $(p_i, \theta_i)$  to point  $p_{i+1}$  (where  $\theta_1$  is the polar angle of  $\overrightarrow{p_1 p_2}$  and  $\theta_{i>1}$  is the polar angle at  $p_i$  of the path from  $(p_{i-1}, \theta_{i-1})$  to  $p_i$ ); if  $p_i$  and  $p_{i+1}$  are at least distance  $d \geq 4$  apart they are connected by a path of type *CS* whose length is at most  $d + 2\pi - 2 \arctan d$ . When any two consecutive points are at least distance 4 apart, this greedy approach yields a path whose length is less than 1.91 times that of the optimum. Without lower bound on the distance between consecutive points the approximation factor of the greedy algorithm cannot be bounded. This was addressed in 2000 by Lee et al. [20] who presented a linear-time approximation algorithm for computing a path that is at most 5.03 times longer than the optimal one; we note that when the distance between any two consecutive points is at least  $d > 2$ , the guarantee on the approximation factor improves to  $1 + \frac{2\pi}{d}$  which is less than 2.58 for  $d = 4$ .

The problem of computing shortest paths of bounded curvature through an *unordered* set of points, referred to as *Dubins TSP*, has also been studied [21, 23, 25, 28]; see also [24] for a short survey. Surprisingly, it is only recently that the corresponding decision problem was shown to be NP-hard [23]. All proposed approximation algorithms are based on a discretization of the directions at the via-points. The discretizations are, however, very rough: in essentially all cases, only one direction is chosen at each point. The stochastic version of this problem, in which the  $n$  targets are randomly distributed, has also been studied; see, e.g., [13, 16, 28].

Boissonnat and Lazard [7] also considered the related problem of computing the convex hull of bounded curvature of a set of points, that is the shortest bounded-curvature closed curve that encloses all the points. Here the path does not necessarily pass through every point. This simplifies the problem because it then reduces to computing the polygon of shortest perimeter whose vertices lie, in order, *inside*

the unit disks centered at the vertices of the (regular) convex hull of the input points. Furthermore, the length of this polygon, defined over the Cartesian product of these disks, is shown to be a convex function, thus the minimum is unique and it can be computed by convex optimization.

Our problem is also related to the problem of computing bounded-curvature shortest paths in the presence of polygonal obstacles. Jacobs and Canny [17] proved the existence of a shortest path when there exists a feasible one.<sup>2</sup> They also proved, in parallel with Fortune and Wilfong [14], that such a shortest path consists of a concatenation of Dubins paths joined at points on the boundary of the obstacles. Fortune and Wilfong [14] also presented an exponential-time and space algorithm for deciding the existence of a feasible path between two configurations. A few years later, Reif and Wang [27] showed that the decision problem corresponding to finding a shortest path is NP-hard. Several approximation algorithms were proposed [17, 30, 33]; in particular, Wang and Agarwal [33] presented a  $O(\frac{n^2}{\varepsilon^2} \log n)$ -time algorithm for computing a  $(1+\varepsilon)$ -approximation of a shortest  $\varepsilon$ -robust path (informally, a path is  $\varepsilon$ -robust if it remains feasible after an  $\varepsilon$ -perturbation of the configurations touching the obstacles). It is only very recently that Backer and Kirkpatrick [3] presented the first polynomial-time algorithm that computes a  $(1+\varepsilon)$ -approximation of a shortest path, or reports that there is no path shorter than a given constant  $\ell$  (the complexity is polynomial in terms of the number of polygon vertices, the number of bits of precision used to specify them,  $\varepsilon^{-1}$ , and  $\ell$ ). Shortest or feasible bounded-curvature paths have also been studied inside convex polygons [1], narrow corridors [5], and among obstacles of bounded curvature [2, 8].

Note finally that other models of car-like robots have also been studied. In particular, the Reeds and Shepp model [26], in which both forward and backward motions are allowed, has been extensively studied. Note also that other, and more general, dynamic constraints have been considered, and that Dubins paths have been generalized to the three-dimensional case [31]. We refer to [19] for a recent overview of such path planning problems.

**Paper organization.** After some preliminaries, the next section outlines the proof of our main result (Theorem 1). We then proceed to prove the local convexity of the length function, first between two configurations, in Section 3, then in  $(\mathbb{S}^1)^n$  over a domain called *lemon* (due to its evocative shape in 2D), in Section 4. In Section 5, we prove that any global minimum of  $F$  belongs to a subset of the lemon region, and in Section 6, we show that this subset admits a “bounding box” that remains inside the lemon region and whose connected components are all convex. We then show, in Section 7, that many of these connected components can often be discarded by considering sharp turns, and conclude the proof of our main result. We analyze the complexity of the ellipsoid method applied to our problem in Section 8. In the first part of Section 9, we extend our results to the case where the initial and the final directions are prescribed, and in the second part of the section, we discuss the connection with the problem of computing curvature-constrained shortest paths in the presence of polygonal obstacles.

## 2 Preliminaries and proof outline

We give here some terminology and present a high-level sketch of the proof of our main result. We say that a sequence of points  $p_1, \dots, p_n$  satisfies the  $(D_d)$  condition if every two consecutive points  $p_i$  and  $p_{i+1}$  are at least distance  $d$  apart (the constant  $d$  is specified where needed).

Recall that a bounded-curvature path is a differentiable path whose curvature is defined almost everywhere, and is at most one in our case. A configuration is defined as a pair  $(p, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ . A path goes from a configuration  $(p_s, \theta_s)$  to a configuration  $(p_f, \theta_f)$  if it starts in  $p_s$  and ends in  $p_f$  with the polar angles of its starting and final tangents being  $\theta_s$  and  $\theta_f$ , respectively. A shortest bounded-curvature path between two configurations, without obstacles, is a concatenation of at most three arcs that are circular arcs of unit radius ( $C$ -segments) or line segments ( $S$ -segments) [12]. Between any two configurations, there are at most six such paths, called Dubins paths, and their types are defined as  $CSC$  or  $CCC$  (or a subsequence thereof). We sometimes need to specify whether the path is turning right (clockwise) or left (counterclockwise) on a circle; we then refer to paths of types  $LSR$ , etc. Note that there is always a unique path of type, say  $LSR$ , between two configurations, because the circular arcs are considered to be shorter than  $2\pi$ . We may also specify the length of an arc as an index; for instance, a path of type  $L_\pi SR$  refer to a path that consists of a circular arc of length  $\pi$  turning left, a line segment, and a circular

<sup>2</sup>Note that this is not trivial and actually not true when reversals are allowed [11], that is for the model of Reeds and Shepp [26].



arc turning right. Finally, if the points  $p_s$  and  $p_f$  are at least distance 4 apart, then a shortest Dubins path between any two configurations  $(p_s, \theta_s)$  and  $(p_f, \theta_f)$  is of type *CSC* [10].<sup>3</sup> Moreover, we have the following straightforward property.

**Lemma 2.** *If  $p_s$  and  $p_f$  are at distance  $d \geq 4$ , then the shortest Dubins path between any configurations  $(p_s, \theta_s)$  and  $(p_f, \theta_f)$  has type *CSC* and the line segment has length at least  $\sqrt{(d-2)^2 - 4}$ .*

*Proof.* Let  $s$  be the length of the straight line segment and  $\lambda$  be the distance between the center of the circles supporting the two circular arcs of the *CSC* path. Either the line segment is an outer tangent to these two circles, in which case  $s = \lambda$ , or it is an inner tangent, in which case we get that  $(\frac{s}{2})^2 + 1 = (\frac{\lambda}{2})^2$  by considering one of the two triangles induced by the non-simple quadrilateral formed by the two circle centers and the two segment endpoints. Hence,  $s = \lambda$  or  $s = \sqrt{\lambda^2 - 4}$ , thus  $s \geq \sqrt{\lambda^2 - 4}$  in all cases. The result follows since  $\lambda + 2 \geq d$ .  $\square$

We denote by  $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$  the space of angles. In some cases, it will be more convenient to consider angles in  $\mathbb{R}$ , that is to lift  $\mathbb{S}^1$  to some interval of length  $2\pi$  in  $\mathbb{R}$ .

**Proof outline.** We outline here our proof of Theorem 1. Let  $\mathcal{L}(\alpha)$  denote the set of points in  $(\mathbb{S}^1)^n$  whose associated shortest path has all its circular arcs, between every consecutive pair of points  $p_i$  and  $p_{i+1}$ , of length less than  $\alpha$  (we consider all possible shortest paths, if it is not unique). Recall that  $F : (\mathbb{S}^1)^n \rightarrow \mathbb{R}$  is the function that maps a sequence  $(\theta_1, \dots, \theta_n)$  of angles to the length of a shortest curvature-constrained path visiting the configurations  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  in order. Our first result, the cornerstone of this paper, is that, under the  $(D_4)$  condition,  $F$  is locally convex over  $\mathcal{L}(\pi)$ . We first show this for the case of two points (Theorem 6). The generalization to  $n$  points is then straightforward since  $F$  decomposes into

$$F(\theta_1, \dots, \theta_n) = F_1^2(\theta_1, \theta_2) + \dots + F_{n-1}^n(\theta_{n-1}, \theta_n),$$

where  $F_i^{i+1}(\theta_i, \theta_{i+1})$  is the length of the shortest path from  $p_i$  to  $p_{i+1}$  and whose tangents in those points have polar angles  $\theta_i$  and  $\theta_{i+1}$ , respectively. We also prove that any global minimum of  $F$  must belong to  $\mathcal{L}(\frac{3\pi}{4})$  under the  $(D_4)$  condition (Lemma 11).

At this point, since  $F$  is locally convex on an open region containing all its global minima, one could hope that it has a unique global minimum which can be computed by convex optimization. However, the geometry of  $\mathcal{L}(\alpha)$  turns out to be quite complicated. Two main issues are that  $\mathcal{L}(\alpha)$  is, in general, not connected, and that the connected components are not convex, when lifted to  $\mathbb{R}^n$ . In particular, we give an example where  $\mathcal{L}(\pi)$  has  $2^{n-2}$  connected components, each of which contains a local minimum of  $F$  (Corollary 10).

To overcome these issues, we first show that there exists a “nice” region  $\mathcal{D}$ , called a *diamond*, such that  $\mathcal{L}(\frac{3\pi}{4}) \subset \mathcal{D} \subset \mathcal{L}(\pi)$  (Lemma 16) and whose connected components are, once lifted to  $\mathbb{R}^n$ , convex polyhedra defined by  $O(n)$  inequalities each (Lemma 17). The inclusion  $\mathcal{D} \subset \mathcal{L}(\pi)$  and the convexity of these components guarantee that the function  $F$  is (globally) convex on each of these polyhedra. The other inclusion  $\mathcal{L}(\frac{3\pi}{4}) \subset \mathcal{D}$  ensures that  $\mathcal{D}$  contains all the global minima of  $F$ . Hence, all global minima of  $F$  can be computed by minimizing  $F$  over every connected component of  $\mathcal{D}$  over which  $F$  is convex (Proposition 18). Unfortunately, there might still be  $\Theta(2^n)$  such components.

To reduce the search space, consider the map  $\sigma : (\mathbb{S}^1)^n \rightarrow \{-1, +1\}^n$  such that  $(\sigma(\theta_1, \dots, \theta_n))_i$  is  $-1$  if the vector with polar angle  $\theta_i$  belongs to the positive cone of  $\overrightarrow{p_i p_{i-1}}$  and  $\overrightarrow{p_{i+1} p_i}$ , and  $+1$  otherwise. The map  $\sigma$  characterizes the connected components of  $\mathcal{D}$ , and we show that if  $p_i$  is not a sharp turn then  $(\sigma(\theta_1, \dots, \theta_n))_i = 1$  for every global minimum  $(\theta_1, \dots, \theta_n)$  of  $F$  (Lemma 20). This narrows the search to  $2^k$  connected components of  $\mathcal{D}$ , where  $k$  is the number of sharp turns among  $p_1, \dots, p_n$ , and Theorem 1 follows.

If  $k = 0$ , we have that all the global minima of  $F$  belong to a single convex polyhedron on which  $F$  is strictly convex. It follows that  $F$  has a unique global minimum which corresponds to the unique shortest path (by Proposition 5). Furthermore, an analysis of the ellipsoid method yields that we can compute a path whose length is at most  $\varepsilon$  plus the length of the globally shortest path, in  $O(n^4 \log \frac{n}{\varepsilon})$  time (Corollary 24).

<sup>3</sup>Although this is not stated explicitly in [10], Bui et al. show (in §4.3) that if a shortest Dubins path is of type *CCC*, then  $p_f$  lies inside a disk of radius 2 (denoted  $C_H$ ) which contains  $p_s$ . Furthermore, if  $p_f$  lies on the boundary of that disk, then the path is also of type *CSC* with the line segment of length zero.

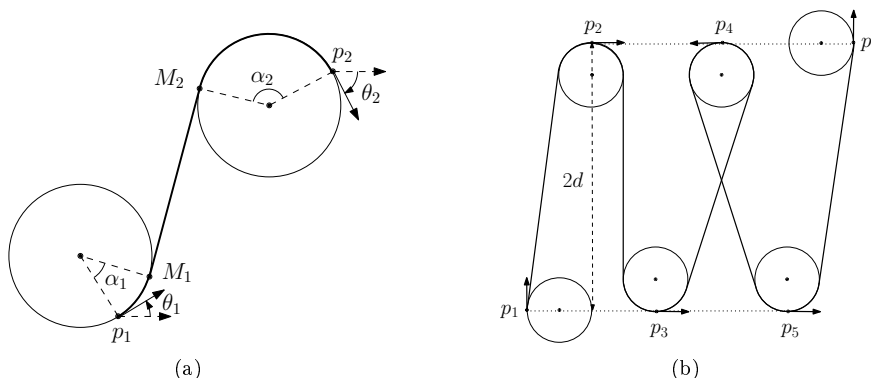


Figure 1: (a) *LSR*-path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$ . (b) Every sequence of polar angles  $\theta_2, \dots, \theta_{n-1}$  in  $\{0, \pi\}$  defines a path that is arbitrarily close to a local optimum, for  $d$  large enough.

### 3 Local convexity of the length function of a *CSC*-path

In this section, we consider the case of two points, and let  $F_{csc}(\theta_1, \theta_2)$  denote the length of a shortest *CSC*-path from a configuration  $(p_1, \theta_1)$  to a configuration  $(p_2, \theta_2)$ .<sup>4</sup> We prove that  $F_{csc}$  is locally strictly convex<sup>5</sup> at any point  $(\theta_1, \theta_2)$  such that both circular arcs of the corresponding path are shorter than  $\pi$ . Figure 2 shows an example of such domain, and the graph of  $F_{csc}(\theta_1, \theta_2)$  over that domain. We emphasize that the local convexity of  $F_{csc}$  holds without any distance assumption although we ignore here *CCC* paths which may be the shortest when  $p_1$  and  $p_2$  are distance less than 4 apart.

We start by showing that for a given path type  $T \in \{LSR, RSL, LSL, RSR\}$ , the length  $F_T(\theta_1, \theta_2)$  of the path of type  $T$  ( $T$ -path) from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is such a locally strictly convex function (Proposition 4). We prove this by computing the Hessian of the length function (Proposition 3). We then prove that this local convexity extends to the length  $F_{csc}(\theta_1, \theta_2) = \min_{T \in \{LSR, RSL, LSL, RSR\}} F_T(\theta_1, \theta_2)$  of a shortest *CSC*-path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  (Theorem 6). We prove this by first showing the interesting property that, if both circular arcs of a *CSC*-path are shorter than  $\pi$ , then this path is *the* shortest *CSC*-path (Proposition 5).

The proofs are rather technical, and, for clarity, we provide here proof sketches and postpone the complete proofs to Appendix A.

**Notation.** Refer to Figure 1(a). For a given path type  $T \in \{LSR, RSL, LSL, RSR\}$ , let  $F_T(\theta_1, \theta_2)$  denote the length of the  $T$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$ . For a given *CSC*-path, let  $\alpha_i$  be the length of its  $i$ -th circular arc, let  $M_1$  and  $M_2$  be the first and last endpoint of its line segment, and let  $M_1M_2$  denote the Euclidean distance from  $M_1$  to  $M_2$ . Let  $\mu_B$  be equal to 1 if  $B$  is true and to  $-1$  otherwise. In particular, for a type of path  $T \in \{LSR, RSL, LSL, RSR\}$ ,  $\mu_{C_j=R}$  ( $j = 1, 2$ ) is equal to 1 if the type of the  $j$ -th circular arc in  $T$  is  $R$  and it is equal to  $-1$  otherwise. Finally, let  $\delta_{i,j}$  equal to 1 if  $i = j$  and to 0 otherwise.

We start by computing the first and second derivatives of the length function  $F_T$ , and its Hessian. The proof is calculatory and technical but a careful observation of the expressions involved leads to simple expressions of the derivatives.

**Proposition 3.** *For a given path type  $T \in \{LSR, RSL, LSL, RSR\}$ , the length  $F_T(\theta_1, \theta_2)$  of the  $T$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is twice differentiable at any point  $(\theta_1, \theta_2)$  such that the corresponding  $T$ -path exists and none of its arcs vanishes. Furthermore,*

$$\frac{\partial F_T(\theta_1, \theta_2)}{\partial \theta_i} = \mu_{i=1} \mu_{C_i=R} (1 - \cos \alpha_i) \quad \frac{\partial^2 F_T}{\partial \theta_i \partial \theta_j} = \delta_{i,j} \sin \alpha_i + \frac{\sin \alpha_i \sin \alpha_j}{M_1 M_2}$$

<sup>4</sup>For two points ( $n = 2$ ), the length function  $F$  was defined in Section 1 as the length of a shortest curvature-constrained path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$ . When  $d$  is sufficiently small,  $F_{csc}$  and  $F$  differ at points  $(\theta_1, \theta_2)$  such that the shortest path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is (only) realized by a *CCC*-path. However, for  $d \geq 4$ , the shortest paths are always of type *CSC*, and thus  $F_{csc}$  and  $F$  coincide.

<sup>5</sup>The local convexity of  $F_{csc}$  naturally refers to the local convexity of  $F_{csc} \circ \tau$ , where  $\tau$  is the quotient map from  $\mathbb{R}^2$  to  $\mathbb{S}^1 \times \mathbb{S}^1$ ; in other words, the angles are seen in  $\mathbb{R}$ .

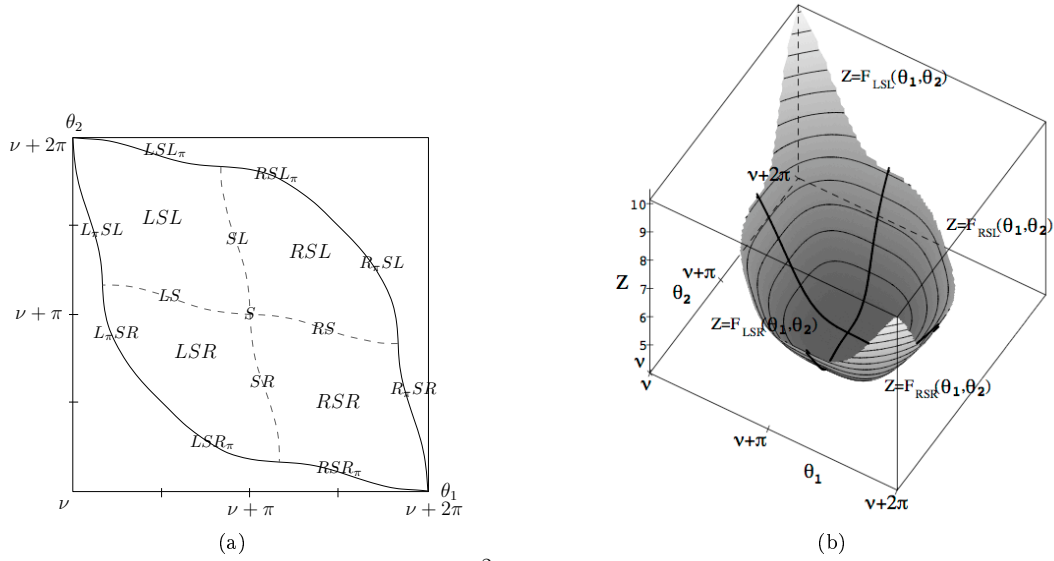


Figure 2: (a) Domain of  $(\theta_1, \theta_2)$  in  $[\nu, \nu + 2\pi]^2$  such that both circular arcs of the shortest  $CSC$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  are shorter than  $\pi$ , where  $\|\overrightarrow{p_1 p_2}\| = 4$  and  $\nu$  is the polar angle of  $\overrightarrow{p_2 p_1}$  (for  $\theta_1 = \theta_2 = \nu + \pi$ , the path is reduced to a line segment). (b) Graph of  $F_{csc}$  over that domain.

and the determinant of the Hessian of  $F_T$  is

$$\sin \alpha_1 \sin \alpha_2 \left( 1 + \frac{\sin \alpha_1 + \sin \alpha_2}{M_1 M_2} \right).$$

If  $(\theta_1, \theta_2)$  is such that the lengths  $\alpha_1$  and  $\alpha_2$  of the circular arcs of the  $T$ -path are in  $(0, \pi)$ , then  $\frac{\partial^2 F_T}{\partial \theta_1^2}$  and the determinant of the Hessian of  $F_T$  are positive. This implies that  $F_T$  is positive definite (by Sylvester's criterion) and thus locally strictly convex at  $(\theta_1, \theta_2)$ . As a consequence, we get the following result.

**Proposition 4.** *For a given path type  $T \in \{LSR, RSL, LSL, RSR\}$ , the length  $F_T(\theta_1, \theta_2)$  of the  $T$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is locally strictly convex at any point  $(\theta_1, \theta_2)$  such that the corresponding  $T$ -path exists, none of its arcs vanishes, and its two circular arcs have length less than  $\pi$ .*

Note that we get trivially a similar result if we consider the length function in terms of only one angle, say  $\theta_1$  with  $\theta_2$  fixed: then  $F_T(\theta_1)$  is locally strictly convex at any  $\theta_1$  such that  $\alpha_1 \in (0, \pi)$ .

**Proposition 5.** *If both circular arcs of a  $CSC$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  are strictly shorter than  $\pi$ , then all the other distinct  $CSC$ -paths are strictly longer.*

*Sketch of proof.* We consider two geometrically distinct paths of type  $T$  and  $T'$  in  $\{LSR, RSL, LSL, RSR\}$ , from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$ , such that both circular arcs of the  $T$ -path are shorter than  $\pi$ . We consider all possible types of  $T$  and  $T'$  in turn and we show using geometric arguments that, in every case, the line segment of the  $T$ -path is shorter than the one of the  $T'$ -path, and that the same holds for the total length of the circular arcs. □

We can now prove the local convexity of the length function  $F_{csc}(\theta_1, \theta_2)$  of the shortest  $CSC$ -paths from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  on the domain of  $(\theta_1, \theta_2)$  such that both circular arcs are shorter than  $\pi$ . Figure 2 shows an example of such domain and of the graph of  $F_{csc}(\theta_1, \theta_2)$  over that domain.

Graph of  $F_{csc}$  over that domain. ( $|p_1 p_2| = 4$  and  $\nu$  is the polar angle of  $\overrightarrow{p_2 p_1}$ ).

**Theorem 6.** *The length  $F_{csc}(\theta_1, \theta_2)$  of the shortest  $CSC$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is locally strictly convex at any point  $(\theta_1, \theta_2)$  such that both circular arcs of the corresponding path are strictly shorter than  $\pi$ . Furthermore,  $F_{csc}$  is  $C^2$  at such a point.*

*Sketch of proof.* Propositions 4 and 5 yield that  $F_{csc}(\theta_1, \theta_2)$  is locally strictly convex at any point such that both circular arcs of the shortest  $CSC$ -path have length in  $(0, \pi)$ . When one circular arc vanishes, two  $T$  and  $T'$ -paths coincide ( $T \neq T'$  in  $\{LSR, \dots\}$ ), and Proposition 3 yields that  $F_{csc}$  is locally  $C^2$  and thus locally convex at this point. Finally, we prove the *strict* local convexity at such a point by considering the third derivatives of the length functions of the  $T$  and  $T'$ -paths.  $\square$

## 4 Local convexity of $F$ and lemon regions

Theorem 6 suggests that the length function  $F(\theta_1, \dots, \theta_n)$  is well-behaved on a certain domain of  $(\mathbb{S}^1)^n$ . In this section, we define this domain and analyze its geometric structure.

**Definition of the lemons.** Assume that condition  $(D_4)$  holds, and let  $\alpha \in (0, \pi]$ . Let  $L_i^{i+1}(\alpha)$  denote the set of angles  $(\theta_i, \theta_{i+1})$  in  $(\mathbb{S}^1)^2$  such that both circular arcs of the shortest path from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  have length (strictly) less than  $\alpha$ . This set is well-defined because Lemma 2 ensures that the shortest path is of type  $CSC$  and Proposition 5 guarantees it is unique (since  $\alpha \leq \pi$ ). Note that  $L_i^{i+1}(\alpha)$  is an open set, since the circular arcs must be *strictly* shorter than  $\alpha$ . Theorem 6 now simply asserts that the length function  $F_i^{i+1}$  is locally strictly convex on  $L_i^{i+1}(\pi)$ . We call  $L_i^{i+1}(\alpha)$  a *lemon* region due to its evocative shape (see Figure 2(a)).

We now define the  $n$ -dimensional lemon region  $\mathcal{L}(\alpha) \subset (\mathbb{S}^1)^n$  as the set of tuples  $(\theta_1, \dots, \theta_n)$  such that the shortest path visiting the configurations  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  has all its circular arcs, between any two consecutive points  $p_i$  and  $p_{i+1}$ , of length less than  $\alpha$ . The shortest path through a sequence of configurations  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  is the concatenation of the shortest paths from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  for  $i = 1, \dots, n-1$ . This ensures, with Proposition 5, that  $\mathcal{L}(\alpha)$  is well-defined for any  $\alpha \in (0, \pi]$ . That also implies that a point  $(\theta_1, \dots, \theta_n)$  is in  $\mathcal{L}(\alpha)$  if and only if, for  $i = 1, \dots, n-1$ , the shortest path from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  uses circular arcs of length less than  $\alpha$ , that is  $(\theta_i, \theta_{i+1}) \in L_i^{i+1}(\alpha)$ . This rewrites as

$$\mathcal{L}(\alpha) = \bigcap_{i=1}^{n-1} (\mathbb{S}^1)^{i-1} \times L_i^{i+1}(\alpha) \times (\mathbb{S}^1)^{n-i-1}, \quad (1)$$

with the convention that  $(\mathbb{S}^1)^0 \times A = A \times (\mathbb{S}^1)^0 = A$ . Now, since  $F(\theta_1, \dots, \theta_n) = \sum_{i=1}^{n-1} F_i^{i+1}(\theta_i, \theta_{i+1})$ , and a sum of locally convex functions is locally convex, Theorem 6 yields that  $F$  is locally convex over  $\mathcal{L}(\pi)$ . Furthermore, since  $F_i^{i+1}(\theta_i, \theta_{i+1})$  is locally *strictly* convex over  $L_i^{i+1}(\pi)$ , it is easy to show that  $F$  is locally *strictly* convex over  $\mathcal{L}(\pi)$ , even though  $F_i^{i+1}$  is not locally strictly convex as a function of  $(\theta_1, \dots, \theta_n)$ :

**Proposition 7.** *If the  $(D_4)$  condition holds, the length function  $F(\theta_1, \dots, \theta_n)$  is  $C^2$  and locally strictly convex over  $\mathcal{L}(\pi)$ .*

*Proof.* First recall that, as shown in the proof of Theorem 6, the function  $F$  is  $C^2$  over  $\mathcal{L}(\pi)$ . The Hessian of  $F$  is thus defined over  $\mathcal{L}(\pi)$  and we prove that it is positive definite over  $\mathcal{L}(\pi)$ , which implies the result. The Hessian of  $F = \sum_{i=1}^{n-1} F_i^{i+1}(\theta_i, \theta_{i+1})$  is

$$H = \begin{pmatrix} \frac{\partial^2 F_1^2}{\partial \theta_1^2} & \frac{\partial^2 F_1^2}{\partial \theta_1 \partial \theta_2} & 0 & \dots & 0 \\ \frac{\partial^2 F_1^2}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 F_1^2}{\partial \theta_2^2} + \frac{\partial^2 F_2^3}{\partial \theta_2^2} & \frac{\partial^2 F_2^3}{\partial \theta_2 \partial \theta_3} & 0 & \dots & 0 \\ 0 & \frac{\partial^2 F_2^3}{\partial \theta_2 \partial \theta_3} & \frac{\partial^2 F_2^3}{\partial \theta_3^2} + \frac{\partial^2 F_3^4}{\partial \theta_3^2} & \frac{\partial^2 F_3^4}{\partial \theta_3 \partial \theta_4} & 0 & \dots & 0 \\ 0 & 0 & \frac{\partial^2 F_3^4}{\partial \theta_3 \partial \theta_4} & \frac{\partial^2 F_3^4}{\partial \theta_4^2} + \frac{\partial^2 F_4^5}{\partial \theta_4^2} & \frac{\partial^2 F_4^5}{\partial \theta_4 \partial \theta_5} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \frac{\partial^2 F_{n-1}^n}{\partial \theta_{n-1} \partial \theta_n} & \frac{\partial^2 F_{n-1}^n}{\partial \theta_n^2} \end{pmatrix}.$$

Hence, for any  $\Theta = (\theta_1, \dots, \theta_n)$ ,

$$\begin{aligned} \Theta^T H \Theta = & \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}^T \begin{pmatrix} \frac{\partial^2 F_1^2}{\partial \theta_1^2} & \frac{\partial^2 F_1^2}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 F_1^2}{\partial \theta_1 \partial \theta_2} & \frac{\partial^2 F_1^2}{\partial \theta_2^2} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix}^T \begin{pmatrix} \frac{\partial^2 F_2^3}{\partial \theta_2^2} & \frac{\partial^2 F_2^3}{\partial \theta_2 \partial \theta_3} \\ \frac{\partial^2 F_2^3}{\partial \theta_2 \partial \theta_3} & \frac{\partial^2 F_2^3}{\partial \theta_3^2} \end{pmatrix} \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} + \dots \\ & + \begin{pmatrix} \theta_{n-1} \\ \theta_n \end{pmatrix}^T \begin{pmatrix} \frac{\partial^2 F_{n-1}^n}{\partial \theta_{n-1}^2} & \frac{\partial^2 F_{n-1}^n}{\partial \theta_{n-1} \partial \theta_n} \\ \frac{\partial^2 F_{n-1}^n}{\partial \theta_{n-1} \partial \theta_n} & \frac{\partial^2 F_{n-1}^n}{\partial \theta_n^2} \end{pmatrix} \begin{pmatrix} \theta_{n-1} \\ \theta_n \end{pmatrix}. \end{aligned}$$

For any  $\Theta$  in  $\mathcal{L}(\pi)$ ,  $(\theta_i, \theta_{i+1})$  belongs to the lemon  $L_i^{i+1}(\pi)$ , for all  $i = 1, \dots, n-1$ , and Theorem 6 implies that every term of the above sum is strictly positive. Hence  $H$  is positive definite over  $\mathcal{L}(\pi)$  which concludes the proof.  $\square$

**Lifting.** Before we proceed to use the convexity of  $F$  over  $\mathcal{L}(\pi)$ , let us give an explicit lifting of  $(\mathbb{S}^1)^n$  to some hypercube in  $\mathbb{R}^n$  that preserves the connected components of  $\mathcal{L}(\pi)$ . Let  $\nu_{i+1}^i$  denote the polar angles of  $\overrightarrow{p_{i+1}p_i}$  and consider the family  $\mathcal{H}$  of all the hyperplanes

$$\begin{aligned} H_i^+ &= \{(\theta_1, \dots, \theta_n) \in (\mathbb{S}^1)^n \mid \theta_i = \nu_{i+1}^i\}, \quad i = 1, \dots, n-1, \\ H_i^- &= \{(\theta_1, \dots, \theta_n) \in (\mathbb{S}^1)^n \mid \theta_i = \nu_i^{i-1}\}, \quad i = 2, \dots, n. \end{aligned}$$

**Lemma 8.**  $\mathcal{L}(\pi)$  does not intersect any hyperplane from  $\mathcal{H}$ .

*Proof.* In any CSC-path from  $(p_i, \nu_{i+1}^i)$  to  $(p_{i+1}, \theta_{i+1})$ , for any  $\theta_{i+1} \in \mathbb{S}^1$ , the circular arc following  $p_i$  has length at least  $\pi$  if  $|p_i p_{i+1}| \geq 2$ . Thus,  $\mathcal{L}(\pi)$  does not intersect any hyperplane  $H_i^+$ , and similarly for  $H_i^-$ .  $\square$

For  $i = 1, \dots, n-1$ , let  $\Lambda_i$  be the interval (closed on its left side and open on its right side) of length  $2\pi$  that contains 0 and has its endpoints in  $\nu_{i+1}^i + 2\pi\mathbb{Z}$ , and let  $\Lambda_n = \Lambda_{n-1}$ . Now, let

$$\Lambda = \prod_{1 \leq i \leq n} \Lambda_i \subset \mathbb{R}^n.$$

For  $1 < i < n$ ,  $\Lambda_i$  contains one point from  $\nu_i^{i-1} + 2\pi\mathbb{Z}$  which splits it into two intervals; we denote the larger of these intervals by  $\Lambda_i^+$  and the smaller by  $\Lambda_i^-$  (if the two intervals have the same length the names have no importance); by convention we let  $\Lambda_1^+ = \Lambda_1^- = \Lambda_1$  and  $\Lambda_n^+ = \Lambda_n^- = \Lambda_n$ . Now, in the lifting of  $(\mathbb{S}^1)^n$  to  $\Lambda$ , each cell of  $(\mathbb{S}^1)^n \setminus \mathcal{H}$  is lifted to a box  $\prod_{1 \leq i \leq n} \Lambda_i^{\varepsilon_i}$  where  $\varepsilon_i \in \{-, +\}$ . An immediate consequence of Lemma 8 is that the image, through this lifting of every connected component of  $\mathcal{L}(\pi)$  is connected.

**Three issues.** With Proposition 7, one could hope to use convex optimization methods to find the minimum of the length function  $F$ . This can only work if  $\mathcal{L}(\pi)$  contains a global minimum of  $F$ . We show in Section 5 that under the  $(D_4)$  condition *any* global minimum of  $F$  lies in  $\mathcal{L}(\frac{3\pi}{4}) \subset \mathcal{L}(\pi)$  (Lemma 11). This still leaves us with two issues.

On the one hand, it is clear from the example of Figure 2(a) that even for two points,  $\mathcal{L}(\pi)$  may be non-convex; this means that there could be many (local) minima of  $F$  in every connected component of  $\mathcal{L}(\pi)$ . We handle this in Section 6 by describing a simple region  $\mathcal{D}$ , which we call a *diamond*, such that  $\mathcal{L}(\frac{3\pi}{4}) \subset \mathcal{D} \subset \mathcal{L}(\pi)$  and the lifting  $(\mathbb{S}^1)^n \rightarrow \Lambda$  maps each connected component of  $\mathcal{D}$  to a convex polyhedron.

On the other hand, the regions  $\mathcal{L}(\pi)$ ,  $\mathcal{L}(\frac{3\pi}{4})$  and  $\mathcal{D}$  may have exponentially many connected components. Indeed, by Lemma 8,  $\mathcal{L}(\pi)$  is contained in the interior of the cells of the arrangement of  $\mathcal{H}$ , and Figure 1(b) shows an example of via-points for which (for  $d$  large enough) every choice of  $(\theta_1, \dots, \theta_n)$  in  $\{\frac{\pi}{2}\} \times \{0, \pi\}^{n-2} \times \{\frac{\pi}{2}\}$  belongs to  $\mathcal{L}(\frac{3\pi}{4})$  and to a distinct cell of that arrangement. In section 7 we give a geometric condition on  $(p_{i-1}, p_i, p_{i+1})$  that implies that for  $1 < i < n$ , the polar angle of the tangent at  $p_i$  to any globally shortest path belongs to  $\Lambda_i^+$  (Lemma 20); this will avoid searching in each of the  $2^{n-2}$  cells of  $(\mathbb{S}^1)^n \setminus \mathcal{H}$ .

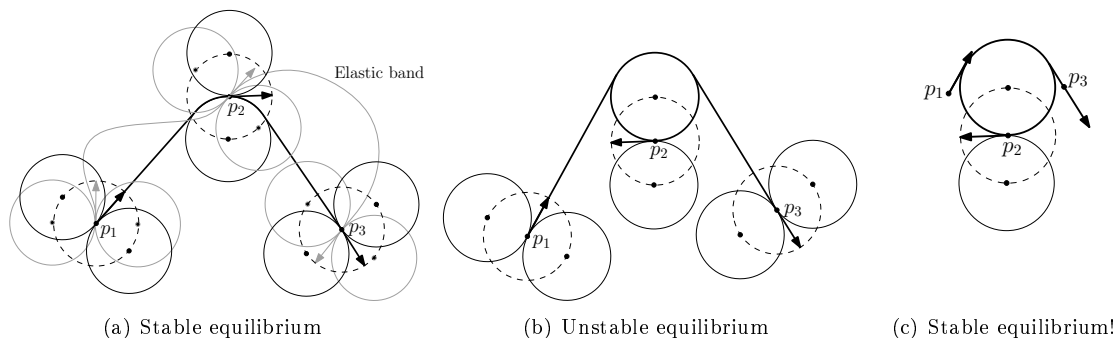


Figure 3: Bounded-curvature paths seen as equilibria of a mechanical device consisting of freely-rotating pulleys attached at  $p_1, p_2, p_3$  and an elastic band.

## 5 Any global minimum of $F$ belongs to $\mathcal{L}(\frac{3\pi}{4})$

We prove here that not only does any global minimum of the length function  $F$  belong to the region  $\mathcal{L}(\pi)$ , but it also belongs to the smaller region  $\mathcal{L}(\frac{3\pi}{4})$ . Let  $\gamma$  be a shortest bounded-curvature path through a sequence of configurations  $(p_1, \tilde{\theta}_1), \dots, (p_n, \tilde{\theta}_n)$ . We assume that the points  $p_1, \dots, p_n$  satisfy condition  $(D_4)$  so that  $\gamma$  is of type *CSC* between any two consecutive configurations (by Lemma 2).

We first consider locally shortest paths, where  $\gamma$  is *locally shortest* if it cannot be shortened by perturbing the  $\theta_i$ , that is if  $\tilde{\Theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_n)$  is a local minimum of  $F$ . Locally shortest paths can be nearly characterized as follows (we postpone the complete proof to section 5.1):

**Proposition 9.** *If  $\gamma$  is a locally shortest path, then (i) its initial and final circular arcs vanish, (ii) the two circular arcs preceding and following every point  $p_i$ ,  $1 < i < n$ , have the same orientation (*R* or *L*), and (iii) their lengths are either equal or sum up to  $2\pi$ .*

*Conversely, if  $\gamma$  satisfies (i) and (ii), and if (iii)' the lengths of the two circular arcs incident to every  $p_i$ ,  $1 < i < n$ , are equal and strictly less than  $\pi$ , then  $\gamma$  is a locally shortest path; on the other hand, if this length is strictly larger than  $\pi$  for some  $p_i$ , then  $\gamma$  is not locally shortest if  $(D_{2+\sqrt{5}})$  holds, and  $\gamma$  might be locally shortest otherwise.<sup>6</sup>*

*Sketch of proof.* The first two claims follow from the expression of the first derivatives of the length of a *CSC*-path obtained in Proposition 3, and from the local convexity of the length function  $F$  of  $\gamma$  (Proposition 7). Concerning the last claim, Figures 3(b) and 3(c) show two configurations where the path has circular arcs of length more than  $\pi$ ; in the former the path is not locally shortest, whereas it is in the latter. This follows from considerations on the length of the segments. If  $(D_{2+\sqrt{5}})$  holds, then the segments are longer than 1 (by Lemma 2), and Proposition 3 yields that  $\frac{\partial^2 F(\tilde{\Theta})}{\partial \theta_i^2}$  is strictly negative, which shows that  $\tilde{\Theta}$  is not a local minimum of  $F$ . Conversely, a third-order Taylor expansion of the length function reveals that when one of the line segments is sufficiently short,  $\gamma$  is locally shortest.  $\square$

Note that these properties are intuitively obvious if one consider a mechanical model where the path is modeled by an elastic band passing through the points  $p_1, \dots, p_n$  at which freely rotating double-pulleys (i.e., two tangent unit disks) are attached (see Figure 3 and [9]). It is, however, interesting to note the limitation of this intuition: if the two circular arcs before and after every point  $p_i$  have the same length, then the mechanical model is at an equilibrium; if these arcs are shorter than  $\pi$ , it seems clear that this equilibrium is stable, implying that the path is locally shortest, and Proposition 9 indeed proves it; however, if these circular arcs are strictly longer than  $\pi$ , it also seems fairly clear that the equilibrium is unstable, implying that the path is not locally shortest, but Proposition 9 shows that this is not necessary true and it depends on the length of the line segments.

<sup>6</sup>We do not claim that the bound  $2 + \sqrt{5}$  is sharp in the constraint  $(D_{2+\sqrt{5}})$ .

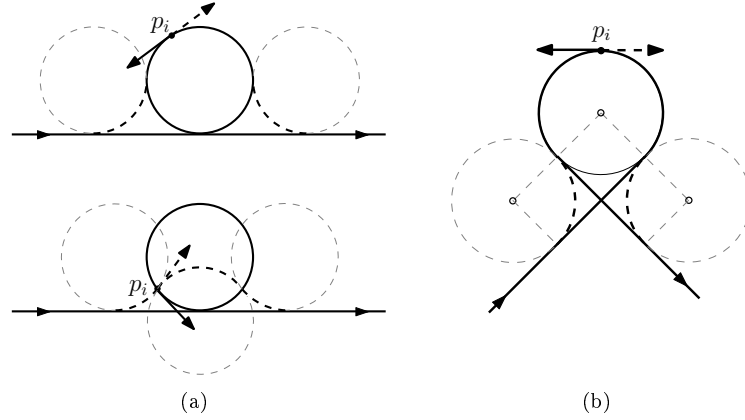


Figure 4: A globally shortest path has all its circular arcs of length at most  $\frac{3\pi}{4}$  (if  $(D_{2+2\sqrt{2}})$  holds).

**Corollary 10.** *There might be at least  $2^{n-2}$  locally shortest paths of bounded curvature through  $p_1, \dots, p_n$ .*

*Proof.* This lower bound follows from the second statement of Proposition 9. Indeed, refer to Figure 1(b), and consider  $n$  points with coordinates  $(2i, (-1)^i d)$  for some constant  $d$ , and the polar angles  $\theta_1 = \frac{\pi}{2}$ ,  $\theta_i = 0$  or  $\pi$  for  $1 < i < n$ , and  $\theta_n = (-1)^n \frac{\pi}{2}$ . If  $d$  is sufficiently large, then any of the  $2^{n-2}$  choices of paths through  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  depicted in the figure are arbitrarily close to locally shortest paths.  $\square$

We can now prove the main result of this section. We first give a simple proof of this result under the  $(D_{2+2\sqrt{2}})$  condition, and postpone the complete proof, under the  $(D_4)$  condition, to Section 5.2.

**Lemma 11.** *Under condition  $(D_4)$ , in any globally shortest path  $\gamma$  the circular arcs preceding and following each via-point have length less than  $\frac{3\pi}{4}$ .*

*Sketch of proof.* We give here a simple proof of the lemma under the  $(D_{2+2\sqrt{2}})$  condition. Let  $C_i^-$  and  $C_i^+$  denote the circular arcs of  $\gamma$  that precede and follow  $p_i$ , respectively. By Proposition 9, since  $\gamma$  is a locally shortest path (and  $2 + 2\sqrt{2} > 2 + \sqrt{5}$ ), the length of  $C_i^-$  and  $C_i^+$  are equal and less than  $\pi$ , or their lengths sum up to  $2\pi$ . By Lemma 2,  $(D_{2+2\sqrt{2}})$  implies that the line segments preceding and following  $p_i$  have length at least 2; the path can thus be trivially shortened if  $C_i^-$  and  $C_i^+$  sum up to  $2\pi$  (see Figure 4(a)), contradicting the global optimality of  $\gamma$ . On the other hand, if the circular arcs have equal length in  $[\frac{3\pi}{4}, \pi]$  (see Figure 4(b)), the fact that the segments preceding and following  $p_i$  have length at least 2 again allows to shorten the path (by convexity of the two circular shortcuts), and the statement follows.  $\square$

## 5.1 Proof of Proposition 9

Recall that  $\gamma$  is a locally shortest path that visit the configurations  $(p_1, \tilde{\theta}_1), \dots, (p_n, \tilde{\theta}_n)$  in order, i.e.,  $\tilde{\Theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_n)$  is a local minimum of  $F$ , and  $\gamma$  is a shortest Dubins path between every two consecutive configurations. We prove Proposition 9, which partially characterizes the geometry of  $\gamma$  under the  $(D_4)$  condition (which ensures that  $\gamma$  is a concatenation of type *CSC*-paths). This characterization is, unfortunately, not complete even in the case where condition  $(D_{2+\sqrt{5}})$  holds. To clarify the subtlety of the statement, we restate Proposition 9 in terms of the following properties:

- (a) The initial and final circular arcs of  $\gamma$  vanish.
- (b) The circular arcs that precede and follow  $p_i$  have the same orientation (R or L), for  $1 < i < n$ .
- (c<sub>i</sub>) The circular arcs that precede and follow  $p_i$  have same length.
- (c<sub>i</sub><sup>+</sup>) The circular arcs that precede and follow  $p_i$  have same length, which is strictly less than  $\pi$ .
- (d<sub>i</sub>) The lengths of the circular arcs that precede and follow  $p_i$  sum up to exactly  $2\pi$ .

Then, Proposition 9 can be rewritten as follows:

$$\begin{aligned}
\gamma \text{ is locally shortest} &\Rightarrow (a) \text{ and } (b) \text{ and, } \forall 1 < i < n, \quad (c_i) \text{ or } (d_i) \\
\gamma \text{ is locally shortest} &\Leftarrow (a) \text{ and } (b) \text{ and, } \forall 1 < i < n, \quad (c_i^+) \\
(D_{2+\sqrt{5}}) \text{ and } \gamma \text{ is locally shortest} &\Rightarrow (a) \text{ and } (b) \text{ and, } \forall 1 < i < n, \quad (c_i^+) \text{ or } (d_i) \\
\gamma \text{ is locally shortest} &\nRightarrow (a) \text{ and } (b) \text{ and, } \forall 1 < i < n, \quad (c_i^+) \text{ or } (d_i).
\end{aligned}$$

The following three lemmas prove these four statements. In the proofs, we denote by  $C_i^-$  (resp.  $C_i^+$ ) the circular arc preceding (resp. following)  $p_i$ , and by  $\alpha_i^-$  (resp.  $\alpha_i^+$ ) the length of this arc.

**Lemma 12.** *If  $\gamma$  is a locally shortest path, then (a) its initial and final circular arcs vanish, (b) the two circular arcs preceding and following every point  $p_i$  ( $1 < i < n$ ) have the same orientation (R or L), and their lengths are either  $(c_i)$  equal or  $(d_i)$  sum up to  $2\pi$ .*

*Proof.* Since  $F$  is the minimum of several length functions (associated with different path types), it is difficult to determine where  $F$  is differentiable (we only know that  $F$  is differentiable over  $\mathcal{L}(\pi)$ , by Theorem 6). We thus consider, in the proof, the length function associated with the path type of  $\gamma$ , instead of  $F$ .

Recall that Proposition 3 states that the length of a CSC-path of type  $T$  from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is differentiable at any  $(\theta_1, \theta_2)$  such that the circular arcs of the corresponding  $T$ -path do not vanish, and for  $i = 1, 2$ :

$$\frac{\partial F_T(\theta_1, \theta_2)}{\partial \theta_i} = \mu_{i=1} \mu_{C_i=R} (1 - \cos \alpha_i).$$

This first implies Statement (a). Indeed, consider the subpath of  $\gamma$  between  $p_1$  and  $p_2$ , let  $T$  be its type, and suppose for a contradiction that the circular arc at  $p_1$  does not vanish. If the circular arc at  $p_2$  does not vanish either, then  $F_T$  is differentiable at  $\tilde{\theta}_1$  and its derivative,  $\mu_{C_1=R} (1 - \cos \alpha_1)$ , is nonzero; thus  $\gamma$  is not locally shortest, a contradiction. On the other hand, if the circular arc at  $p_2$  vanishes, the type  $T$  is not uniquely defined, but it can be chosen so that the path changes continuously if  $\theta_1$  increases from  $\tilde{\theta}_1$  (and, similarly, if  $\theta_1$  decreases); the length of the path thus changes continuously and may decrease since the derivative defined by continuity at  $\tilde{\theta}_1$  is nonzero and does not depend on the orientation  $L$  or  $R$  at  $p_2$ . Hence, the initial arc of  $\gamma$  vanishes, and similarly for its final arc.

We now prove the rest of the lemma. Consider any nonterminal point  $p_i$ , and the subpath of  $\gamma$  between  $p_{i-1}$  and  $p_{i+1}$ ; denote  $\gamma_i$  this subpath. Let  $F_{\mathcal{T}}(\theta_i)$  be the length of the path from  $(p_{i-1}, \tilde{\theta}_{i-1})$ , through  $(p_i, \theta_i)$ , and to  $(p_{i+1}, \tilde{\theta}_{i+1})$ , whose type before and after  $p_i$  is that of  $\gamma_i$  (these types are not uniquely defined if some circular arc vanishes). Proposition 3 then yields that if the circular arcs of the subpath of  $\gamma$  do not vanish, then  $F_{\mathcal{T}}(\theta_i)$  is differentiable at  $\tilde{\theta}_i$ , and

$$F'_{\mathcal{T}}(\tilde{\theta}_i) = -\mu_{C_i^-=R} (1 - \cos \alpha_i^-) + \mu_{C_i^+=R} (1 - \cos \alpha_i^+).$$

Since  $\tilde{\theta}_i$  is a local minimum of  $F_{\mathcal{T}}$ , either  $F'_{\mathcal{T}}(\tilde{\theta}_i) = 0$  or  $F_{\mathcal{T}}$  is not differentiable at  $\tilde{\theta}_i$ . In the latter case, some circular arcs of  $\gamma_i$  vanishes, and the types of the CSC-path before and after  $p_i$  can then be chosen so that the corresponding path from  $(p_{i-1}, \tilde{\theta}_{i-1})$ , through  $(p_i, \theta_i)$ , and to  $(p_{i+1}, \tilde{\theta}_{i+1})$  changes continuously, and so its length, when  $\theta_i$  increases from  $\tilde{\theta}_i$  (and, similarly, if  $\theta_i$  decreases). Furthermore, the value of the derivative of  $F_{\mathcal{T}}$  defined by continuity at  $\tilde{\theta}_i$  is independent of that choice of type (since  $\mu_{C_i^\pm=R} (1 - \cos \alpha_i^\pm) = 0$  when  $C_i^\pm$  vanishes). Hence, if the derivative is negative, the length of the path decreases when  $\theta_i$  increases from  $\tilde{\theta}_i$ , contradicting its optimality (and, similarly, if the derivative is positive). Therefore,  $F'_{\mathcal{T}}(\tilde{\theta}_i) = 0$  in all cases.

Now, if the orientations (R or L) of the two circular arcs  $C_i^-$  and  $C_i^+$  differ,  $F'_{\mathcal{T}}(\tilde{\theta}_i) = \mu_{C_i^+=R} (2 - \cos \alpha_i^- - \cos \alpha_i^+)$  which is zero only if  $\alpha_i^- = \alpha_i^+ = 0$ ; in that case, the arcs may be considered to have the same orientation, which implies Statement (b). It follows that  $F'_{\mathcal{T}}(\theta_i) = \mu_{C_i^+=R} (\cos \alpha_i^- - \cos \alpha_i^+)$ , which is zero only if  $\alpha_i^- = \alpha_i^+$  or  $\alpha_i^- + \alpha_i^+ = 2\pi$  modulo  $2\pi$ ; moreover these equalities are true not modulo  $2\pi$  since  $0 \leq \alpha_i^\pm < 2\pi$ , which proves Statement  $(c_i-d_i)$ .  $\square$

**Lemma 13.**  *$\gamma$  is a locally shortest path if (a) its initial and final circular arcs vanish, (b) the two circular arcs preceding and following every point  $p_i$  ( $1 < i < n$ ) have the same orientation (R or L), and  $(c_i^+)$  they have equal length, which is strictly shorter than  $\pi$ .*



*Proof.* From the expression of the first-order derivative of  $F$  (Proposition 3), we have that:

$$\left| \frac{\partial F(\Theta)}{\partial \theta_i} \right| = \begin{cases} 1 - \cos \alpha_1^+ & \text{for } i = 1, \\ \cos \alpha_i^- - \cos \alpha_i^+ & \text{for } 1 < i < n, \\ 1 - \cos \alpha_n^- & \text{for } i = n. \end{cases}$$

Since  $\alpha_1^+ = \alpha_n^- = 0$  and  $\alpha_i^- = \alpha_i^+$  for  $1 < i < n$ , the gradient of  $F$  vanishes at any point  $\Theta$  for which the path  $\gamma$  satisfies the hypotheses of the lemma. These hypotheses also imply, by Proposition 7, that the length function is locally strictly convex in  $\Theta$ , which concludes the proof.  $\square$

**Lemma 14.** *If  $(D_{2+\sqrt{5}})$  holds and  $\gamma$  is a locally shortest path then, for  $1 < i < n$ , the total length of the two circular arcs preceding and following  $p_i$  is smaller or equal to  $2\pi$ . This can be false if  $(D_{2+\sqrt{5}})$  does not hold.*

*Proof.* We start by proving the first statement of the lemma. Consider a locally shortest path  $\gamma$ . By Lemma 12, the lengths  $\alpha_i^+$  and  $\alpha_i^-$  of the two circular arcs incident to  $p_i$  ( $1 < i < n$ ) are either equal or sum up to  $2\pi$ . Assume, for a contradiction, that  $\alpha_i^+ = \alpha_i^- \in (\pi, 2\pi)$  and that the condition  $(D_{2+\sqrt{5}})$  holds (see Figure 3(b)). Proposition 3 yields that for  $1 < i < n$ ,

$$\frac{\partial^2 F(\Theta)}{\partial \theta_i^2} = \sin \alpha_i^- + \frac{\sin^2 \alpha_i^-}{s_{i-1}^{i-1}} + \sin \alpha_i^+ + \frac{\sin^2 \alpha_i^+}{s_i^{i+1}}, \quad (2)$$

where  $s_i^{i+1}$  denote the length of the line segment between  $p_i$  and  $p_{i+1}$  in  $\gamma$ . The condition  $(D_{2+\sqrt{5}})$  ensures that  $s_i^{i+1} \geq 1$  (Lemma 2), thus  $\alpha_i^\pm \in (\pi, 2\pi)$  implies that  $\frac{\partial^2 F(\Theta)}{\partial \theta_i^2} < 0$ , contradicting the local optimality of  $\gamma$ . This proves the first statement.

We now prove the second part of the lemma. We consider a path  $\gamma$  through three points  $p_1, p_2, p_3$  that consists, as in Figure 3(c), of a sufficiently short line segment, two circular arcs of equal length in  $(\pi, 2\pi)$  (and same orientation  $L$  or  $R$ ), and another sufficiently short line segment. Let  $\tilde{\Theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$  be the sequence of polar angles of  $\gamma$  at  $p_1, p_2, p_3$ .

First note that, as in the proof of Lemma 13, the gradient of  $F$  vanishes at  $\tilde{\Theta}$ . Since the circular arcs at  $p_1$  and  $p_3$  vanish, Proposition 3 also yields that all partial second derivatives of  $F$  are zero at  $\tilde{\Theta}$  except for  $\frac{\partial^2 F(\tilde{\Theta})}{\partial \theta_2^2}$  which is strictly positive if the line segments of  $\gamma$  are sufficiently short (see Equation (2)).

Hence, the second-order Taylor expansion of  $F$  at  $\tilde{\Theta}$  is  $F(\tilde{\Theta} + \Theta) = F(\tilde{\Theta}) + \frac{1}{2} \frac{\partial^2 F(\tilde{\Theta})}{\partial \theta_2^2} \theta_2^2 + o(\|\Theta\|^2)$ , and it is strictly greater than  $F(\tilde{\Theta})$  when  $\theta_2 \neq 0$  and  $\|\Theta\|$  is small enough.

When  $\theta_2 = 0$ , we consider the third-order Taylor expansion. However, since the length function  $F$  is not three-times differentiable at  $\tilde{\Theta}$ , we consider the length function  $F_{T_{12}T_{23}} = F_{T_{12}} + F_{T_{23}}$  where  $T_{i,i+1}$  is one of the two *CSC*-types of  $\gamma$  between  $p_i$  and  $p_{i+1}$  (there are two types since one circular arc vanishes), and  $F_{T_{i,i+1}}$  is the length of the shortest path of type  $T_{i,i+1}$  from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$ . Then, when  $\theta_2 = 0$ , the third-order Taylor expansion of  $F_{T_{12}T_{23}}$  at  $\tilde{\Theta}$  is (see Equation (21))  $F_{T_{12}T_{23}}(\tilde{\Theta} + \Theta) = F_{T_{12}T_{23}}(\tilde{\Theta}) + \frac{1}{6} \frac{\partial^3 F_{T_{12}}(\tilde{\Theta})}{\partial \theta_1^3} \theta_1^3 + \frac{1}{6} \frac{\partial^3 F_{T_{23}}(\tilde{\Theta})}{\partial \theta_3^3} \theta_3^3 + o(\|\Theta\|^3)$ . We have proved in the proof of Theorem 37 that, for  $\|\Theta\|$  small enough,  $\frac{\partial^3 F_{T_{12}}(\tilde{\Theta})}{\partial \theta_1^3} \theta_1^3 = \mu_{C_1=R} \theta_1^3$ , which is strictly positive unless  $\theta_1 = 0$ , and similarly for  $\frac{\partial^3 F_{T_{23}}(\tilde{\Theta})}{\partial \theta_3^3} \theta_3^3$ . Hence,  $F_{T_{12}T_{23}}(\tilde{\Theta} + \Theta) > F_{T_{12}T_{23}}(\tilde{\Theta})$  for  $\|\Theta\| > 0$  small enough. Hence, independently of whether  $\theta_2$  vanishes,  $F(\tilde{\Theta} + \Theta) > F(\tilde{\Theta})$  for all  $\|\Theta\| > 0$  sufficiently small, and thus  $\gamma$  is locally shortest. We conclude the proof by noting that our proof of the second statement only requires that one of the two line segment of  $\gamma$  is sufficiently short.  $\square$

## 5.2 Proof of Lemma 11

We give here a complete proof of Lemma 11 which states that, under condition  $(D_4)$ , in any globally shortest path  $\gamma$  the circular arcs preceding and following each via-point have length less than  $\frac{3\pi}{4}$ . We argue that if the length of a circular arc preceding or following  $p_i$  is at least  $\frac{3\pi}{4}$  then the part of the path  $\gamma$  between  $p_{i-1}$  and  $p_{i+1}$  can be transformed into a shorter path that still visits  $p_{i-1}$ ,  $p_i$  and  $p_{i+1}$  in that order, contradicting its global optimality.

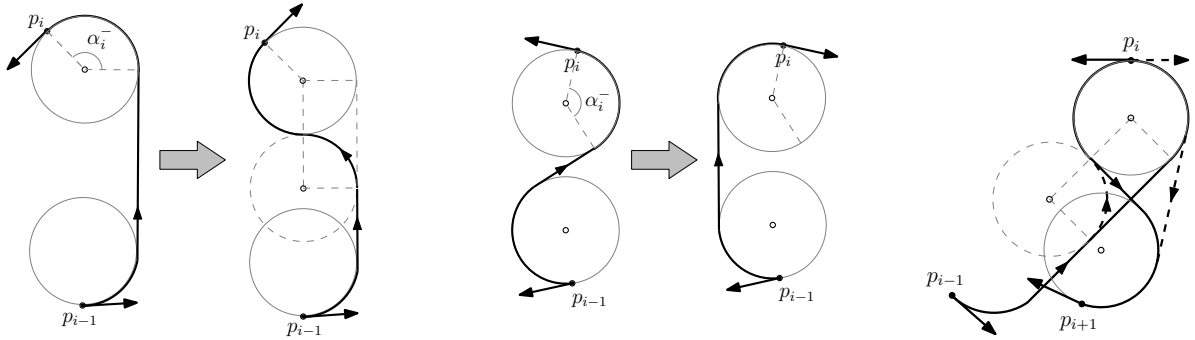


Figure 5: The transformations A (left) and B (center) and a combination of both (right).

**The transformations.** Let  $C_i^-$  and  $C_i^+$  denote the circular arcs of  $\gamma$  that precede and follow  $p_i$ , respectively, let  $\alpha_i^-$  and  $\alpha_i^+$  denote their lengths, and let  $s^-$  and  $s^+$  denote the lengths of the line segments of the path  $\gamma$  that precedes and follows  $p_i$  respectively. We define two elementary transformations on the path from  $p_{i-1}$  to  $p_i$ ; symmetric transformations between  $p_i$  and  $p_{i+1}$  are obtained by considering the path from  $p_{i+1}$  to  $p_i$  (traced backward).

Both transformations are illustrated in Figure 5. If  $s^- \geq 2$ , *transformation A* replaces the *CSC* path from  $(p_{i-1}, \theta_{i-1})$  to  $(p_i, \theta_i)$  by a *CSCC* from  $(p_{i-1}, \theta_{i-1})$  to  $(p_i, \theta_i + \pi)$ . If  $s^- < 2$ , the  $(D_4)$  condition implies that the path from  $p_{i-1}$  to  $p_i$  is of type *LSR* or *RSL*; *transformation B* then replaces it by a path of type *LSL* or *RSR*, respectively, using the same circles.

Both transformations change the configuration at  $p_i$  from  $(p_i, \theta_i)$  into  $(p_i, \theta_i + \pi)$ , while leaving the configurations in  $p_{i-1}$  and  $p_{i+1}$  unchanged. We make three claims:

- (i) If  $s^- \geq 2$  and  $\frac{3\pi}{4} \leq \alpha_i^- \leq \frac{3\pi}{2}$  then transformation A shortens the path from  $p_{i-1}$  to  $p_i$ .
- (ii) If  $s^- < 2$  and  $\frac{3\pi}{4} \leq \alpha_i^-$  then transformation B shortens the path from  $p_{i-1}$  to  $p_i$ .
- (iii) If  $s^- < 2$  then  $\alpha_i^- \leq \frac{3\pi}{2}$ .

Symmetric claims apply to the transforms between  $p_{i+1}$  and  $p_i$ , under similar conditions on  $\alpha_i^+$  and  $s^+$ . Note that Claim (i) is straightforward, because, if  $\alpha_i^- \leq \frac{3\pi}{2}$ , the difference between the lengths of the original subpath and the new one is  $(2 + \alpha_i^-) - (2\pi - \alpha_i^-) = 2(\alpha_i^- - \pi + 1)$ , which is positive if  $\alpha_i^- \geq \frac{3\pi}{4}$ . Claims (ii) and (iii) are less straightforward and will be proved later.

Now, since  $\gamma$  is a locally shortest path, it must satisfy condition (iii) of Proposition 9, that is  $\alpha_i^- = \alpha_i^+$  or  $\alpha_i^- + \alpha_i^+ = 2\pi$ . We treat the two cases separately.

**Case where  $\alpha_i^- = \alpha_i^+$ .** We can assume that the lengths of *both* circular arcs are at least  $\frac{3\pi}{4}$ . If  $s^-$  is at least 2 then we apply transformation A between  $p_{i-1}$  and  $p_i$ , otherwise we apply transformation B; we proceed similarly between  $p_{i+1}$  and  $p_i$ , obtaining a path from  $(p_{i-1}, \theta_{i-1})$  to  $(p_{i+1}, \theta_{i+1})$  through  $(p_i, \theta_i + \pi)$ .

If  $s^-$  and  $s^+$  are smaller than 2, then Claim (ii) implies that the two transformations shorten the path  $\gamma$ . If exactly one of the two line segments is shorter than 2, then the circular arc, say  $C_i^-$ , of the corresponding *CSC* subpath has length at most  $\frac{3\pi}{2}$  by Claim (iii), and thus the other arc  $C_i^+$  is also shorter than or equal to  $\frac{3\pi}{2}$ . Hence, the two transformations also shorten the path  $\gamma$ . The remaining case is when both  $s^-$  and  $s^+$  are larger than or equal to 2. The proof of Lemma 14 implies that if both segments are longer than 1 then  $\alpha_i^- = \alpha_i^+ \leq \pi$ ; that  $(D_{2+\sqrt{5}})$  does not hold is not an issue here, as its only purpose in the proof of Lemma 14 was to bound from below the length of the segments. Hence,  $\alpha_i^- = \alpha_i^+ < \frac{3\pi}{2}$  and the above transformations again shorten the path  $\gamma$ , and thus concluding the proof when  $\alpha_i^- = \alpha_i^+$ .

**Case where  $\alpha_i^- + \alpha_i^+ = 2\pi$ .** Let  $\lambda^-$  (resp.  $\lambda^+$ ) denote the distance between the centers of the two circles supporting the circular arcs of the path  $\gamma$  between  $p_{i-1}$  and  $p_i$  (resp.  $p_i$  and  $p_{i+1}$ ).

First, if both  $s^-$  and  $s^+$  are larger than or equal to 2, the path can be trivially shortened as shown in Figure 4(a). Second, if both line segments are shorter than 2, we apply transformation B between  $p_{i-1}$

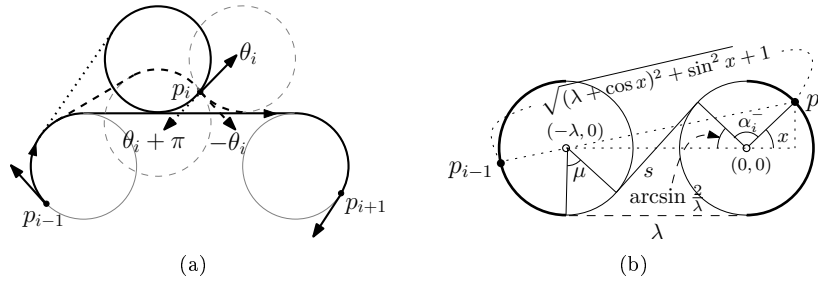


Figure 6: (a) The case where  $\alpha_i^- + \alpha_i^+ = 2\pi$ ,  $s^- < 2$ ,  $s^+ \geq 2$ , and  $\alpha_i^+ > \frac{3\pi}{2}$ . The dashed curve shortens the original path. (b) For the proof of Claims (ii) and (iii) in Section 5.2.

and  $p_i$  and between  $p_i$  and  $p_{i+1}$ . The difference between the lengths of the original subpath and the new one is  $(2\alpha_i^- - \pi + s^- - \lambda^-) + (2\alpha_i^+ - \pi + s^+ - \lambda^+)$ , as shown below in the proof of Claim (ii). Since  $\alpha_i^- + \alpha_i^+ = 2\pi$ , this difference is equal to  $(\pi + s^- - \lambda^-) + (\pi + s^+ - \lambda^+)$ , and  $s^2 = \lambda^2 - 4$  implies that  $\pi + s^\pm > \lambda^\pm$  which concludes the proof when both line segments are shorter than 2.

We are thus left with the case where exactly one line segment has length at least 2; assume, without loss of generality, that  $s^- < 2$  and  $s^+ \geq 2$ . If  $\alpha_i^+ \leq \frac{3\pi}{2}$ , then applying transformation B from  $p_{i-1}$  to  $p_i$  and transformation A from  $p_i$  to  $p_{i+1}$  shorten the path, even if  $\alpha_i^-$  or  $\alpha_i^+$  is shorter than  $\frac{3\pi}{4}$ ; indeed, the difference between the lengths of the original subpath and the new one is  $(2\alpha_i^- - \pi + s^- - \lambda^-) + (2\alpha_i^+ - 2\pi + 2)$ . Since  $\alpha_i^- + \alpha_i^+ = 2\pi$ , this difference is equal to  $\pi + s^- - \lambda^- + 2$  which is positive since, as we just noticed,  $\pi + s^- > \lambda^-$ .

We can thus assume that  $s^- < 2$ ,  $s^+ \geq 2$ , and  $\alpha_i^+ > \frac{3\pi}{2}$  (Figure 6(a)); furthermore, assume without loss of generality, that the line segments lie on the  $x$ -axis, that the arcs  $C_i^\pm$  are oriented  $L$ . Then,  $\theta_i \in [0, \frac{\pi}{2}]$  (since  $C_i^+$  is longer than  $\frac{3\pi}{2}$ ), and the path from  $p_{i-1}$  to  $p_i$  is of type  $RSL$  (since its segment is shorter than 2). As shown in Figure 6(a), we replace the  $RSL$  path from  $(p_{i-1}, \theta_{i-1})$  to  $(p_i, \theta_i)$  by a  $RSR$  path (dashed) from  $(p_{i-1}, \theta_{i-1})$  to  $(p_i, -\theta_i)$ , and we replace the  $LSC$  path from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  by a  $RSC$  path from  $(p_i, -\theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  that uses the same final circular arc. Between  $p_i$  and  $p_{i+1}$ , the difference between the lengths of the original subpath and the new one is larger than  $\alpha_i^+ - (2\pi - \alpha_i^+) = 2\alpha_i^+ - 2\pi$ . Between  $p_{i-1}$  and  $p_i$ , the (dashed) transformed path is shorter than the (dotted)  $RSR$  path from  $(p_{i-1}, \theta_{i-1})$  to  $(p_i, \theta_i + \pi)$ ; call this latter path the intermediate path.<sup>7</sup> Hence, the difference between the lengths of the original path and the transformed path is larger than the difference between the lengths of the original path and the intermediate path, which is  $2\alpha_i^- - \pi + s^- - \lambda^-$  (see the proof of Claim (ii)). Thus, the difference between the lengths of the original subpath from  $p_{i-1}$  to  $p_{i+1}$  and the transformed path is  $(2\alpha_i^- - \pi + s^- - \lambda^-) + (2\alpha_i^+ - 2\pi) = \pi + s^- - \lambda^-$ , which is positive as noted above. It remains to prove Claims (ii) and (iii).

**Proof of Claims (ii) and (iii).** We first argue that the  $(D_4)$  condition and  $s^- < 2$  imply that  $p_{i-1}$  and  $p_i$  lie respectively on the two half-circles shown in bold in Figure 6(b). Suppose, without loss of generality, that  $p_i$  lies on the unit circle  $\mathcal{C}_i$  centered at  $(0, 0)$ , and  $p_{i-1}$  lies on the unit circle  $\mathcal{C}_{i-1}$  centered at  $(-\lambda, 0)$ . Let  $x$  denote the polar angle of  $p_i$ , or more precisely a measure in  $[-\pi, \pi]$  of this angle. When  $p_i$  and  $\mathcal{C}_{i-1}$  are fixed, the maximum of the distance  $\|p_{i-1}p_i\|$  is  $\sqrt{(\lambda + \cos x)^2 + \sin^2 x} + 1$  since it is realized when the segment  $p_i p_{i-1}$  contains the center of  $\mathcal{C}_{i-1}$ . See Figure 6(b). By the  $(D_4)$  assumption, this distance is at least 4, which gives  $\cos x \geq \frac{8-\lambda^2}{2\lambda}$ . Since  $s^- < 2$ ,  $\lambda = \sqrt{s^{-2} + 4} \in [2, 2\sqrt{2}]$ , which implies  $\frac{8-\lambda^2}{2\lambda} \in [0, 1]$ . Thus  $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , that is  $p_i$  lies on the bold half-circle of  $\mathcal{C}_i$  in Figure 6(b), and similarly for  $p_{i-1}$ . Furthermore,  $|x| \leq \arccos \frac{8-\lambda^2}{2\lambda}$ .

It follows that  $\alpha_i^- \leq \frac{3\pi}{2}$  (Claim (iii)) and also that the difference between the lengths of the original subpath and the new one is, with the notation of Figure 6(b),  $(\mu + s^- + \alpha_i^-) - (\lambda + \pi + \mu - \alpha_i^-) =$

<sup>7</sup>This is straightforward by a convexity argument. Indeed, the (dotted)  $RSR$  path from  $(p_{i-1}, \theta_{i-1})$  to  $(p_i, \theta_i + \pi)$  lies on the boundary of its convex hull (this follows from the fact, shown in the proof of Claim (iii), that  $p_{i-1}$  and  $p_i$  lie on the opposite half-circles of the circle supporting the original path from  $p_{i-1}$  to  $p_i$ , as shown in Figure 6(b)). The (dashed)  $RSR$  path  $(p_{i-1}, \theta_{i-1})$  to  $(p_i, -\theta_i)$  has the same property, and its convex hull is included in the other convex hull. It follows that latter path is shorter than the former one.

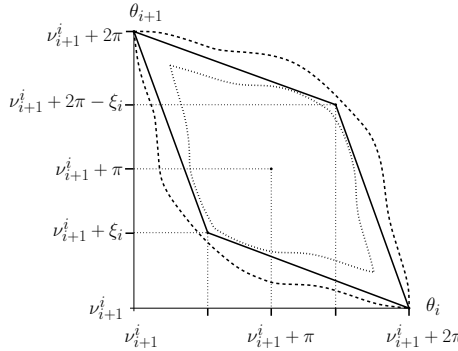


Figure 7: Diamond  $D_i^{i+1}$ , contained in  $L_i^{i+1}(\pi)$  (dashed) and containing  $L_i^{i+1}(\frac{3\pi}{4})$  (dotted) for  $|p_i p_{i+1}| = 4$ .

$2\alpha_i^- - \pi + s^- - \lambda$ . We prove that this difference, denoted  $G$ , is nonnegative. If  $s^- \geq \frac{4}{\pi} - \frac{\pi}{4}$ , then  $\pi s^- + (\frac{\pi}{2})^2 \geq 4$ , and  $(s^- + \frac{\pi}{2})^2 \geq s^{-2} + 4 = \lambda^2$ ; thus  $s^- - \lambda \geq -\frac{\pi}{2}$  and  $G \geq 2\alpha_i^- - \frac{3\pi}{2}$  which is nonnegative when  $\alpha_i^- \geq \frac{3\pi}{4}$ .

Now, suppose that  $s^- < \frac{4}{\pi} - \frac{\pi}{4}$ , which implies that  $\lambda = \sqrt{s^{-2} + 4} \in [2, \frac{4}{\pi} + \frac{\pi}{4}]$ . Consider first the case where  $x \geq 0$  and refer to Figure 6(b). Then  $\alpha_i^- = \pi - \arccos \frac{2}{\lambda} - x \geq \pi - \arccos \frac{2}{\lambda} - \arccos \frac{8-\lambda^2}{2\lambda}$ , and thus  $G \geq \pi - 2\arccos \frac{2}{\lambda} - 2\arccos \frac{8-\lambda^2}{2\lambda} + \sqrt{\lambda^2 - 4} - \lambda$ .  $G$  is positive for  $\lambda \in [2, \frac{4}{\pi} + \frac{\pi}{4}]$  because the right-hand side of this inequality, denoted  $E$ , is positive on that interval; indeed, the derivative of  $E$  with respect to  $\lambda$  is always negative<sup>8</sup> and  $E$  is positive<sup>9</sup> for  $\lambda = \frac{4}{\pi} + \frac{\pi}{4}$ .

For  $x < 0$ ,  $\alpha_i^- \geq \pi - \arccos \frac{2}{\lambda}$ , and, similarly as above,  $G \geq \pi - 2\arccos \frac{2}{\lambda} + \sqrt{\lambda^2 - 4} - \lambda$ . We proved above that the derivative of the right-hand side is always negative, and we prove, similarly as before, that the right-hand side expression is positive<sup>10</sup> for  $\lambda = \frac{4}{\pi} + \frac{\pi}{4}$ . Hence,  $G$  is again positive, which concludes the proof of Claim (ii), and of the lemma.

## 6 Convex bounding boxes of the connected components of $\mathcal{L}(\frac{3\pi}{4})$

In this section we prove that there exists a simple shape, which we call the *diamond*  $\mathcal{D}$ , that contains  $\mathcal{L}(\frac{3\pi}{4})$  and is contained in  $\mathcal{L}(\pi)$ . It will then follow that the lift of  $\mathcal{D}$  to the hypercube  $\Lambda$  defined in Section 4 consists of  $O(2^n)$  convex components, over which  $F$  is convex, and which contain the global minimum of  $F$ .

Consider two consecutive points  $p_i$  and  $p_{i+1}$ , let  $d_i = |p_i p_{i+1}|$  and let  $\xi_i = 2\pi/(d_i - 1/d_i)$ . Recall that  $\nu_{i+1}^i$  denotes the polar angle of the vector  $\overrightarrow{p_{i+1}p_i}$ . We define  $D_i^{i+1}$  as the image of the open quadrilateral with vertices  $(0, 2\pi)$ ,  $(\xi_i, \xi_i)$ ,  $(2\pi, 0)$ , and  $(2\pi - \xi_i, 2\pi - \xi_i)$  under the translation of vector  $(\nu_{i+1}^i, \nu_{i+1}^i)$  (see Figure 7). For two points, this shape has the following property (the proof is technical and is postponed to Appendix B, for clarity):

<sup>8</sup>On one hand, the derivative of  $-2\arccos \frac{8-\lambda^2}{2\lambda}$  is negative (recall that the derivative of  $\arccos f(\lambda)$  is  $-\frac{f'(\lambda)}{\sqrt{1-f^2(\lambda)}}$ ). On the other hand, the derivative of the other terms of  $E$  is equal to  $-\frac{4}{\lambda\sqrt{\lambda^2-4}} + \frac{\lambda}{\sqrt{\lambda^2-4}} - 1$  which is negative or zero if and only if  $\lambda - \frac{4}{\lambda} \leq \sqrt{\lambda^2-4}$ ; the left-hand side is nonnegative because  $\lambda \geq 2$  (by the  $D_4$  assumption), and the inequality is thus equivalent (after squaring) to  $\lambda \geq 2$ , which proves that the derivative of  $E$  is always negative.

<sup>9</sup>The fact that  $E$  is positive for  $\lambda = \frac{4}{\pi} + \frac{\pi}{4}$  can be observed by evaluating numerically the function (which gives roughly 0.25). More formally, this can be proved by first replacing  $-2\arccos z$  by  $-\pi + 2\arcsin z$  and considering the terms of degree up to 9 in the power series of  $\arcsin$ . Since, all the terms in the power series are positive, the resulting expression bounds  $E$  from below. This expression is (up to the positive factor  $\frac{1}{2705829396480\pi^9(16+\pi^2)^9}$ ) a polynomial expression of degree 36 in  $\pi$ , with integer coefficients. The expression can be shown to be positive by regarding it as a polynomial in  $x$ , then by applying the sequence of change of variable  $y = 10x$ ,  $z = y - 31$ ,  $u = 1/z$ , and  $v = u - 1$ , which transforms the interval  $(3.1, 3.2)$  in  $x$  into the interval  $(0, +\infty)$  in  $v$ . All the coefficients of the resulting polynomial are positive, which implies that the initial polynomial in  $x$  is positive over  $(3.1, 3.2)$ , and thus is positive for  $x = \pi$ .

<sup>10</sup>By first replacing  $-2\arccos z$  by  $-\pi + 2\arcsin z$  and using that  $\arcsin z > z$  for  $z > 0$ , we get that  $G \geq \pi + (-\pi + 2\frac{2}{\lambda}) + \sqrt{\lambda^2-4} - \lambda$ , which is equal, for  $\lambda = \frac{4}{\pi} + \frac{\pi}{4} = \frac{16+\pi^2}{4\pi}$ , to  $\frac{16\pi}{16+\pi^2} + (\frac{4}{\pi} - \frac{\pi}{4}) - (\frac{4}{\pi} + \frac{\pi}{4}) = \frac{\pi(16-\pi^2)}{2(16+\pi^2)} > 0$ .

**Lemma 15.** *If  $|p_i p_{i+1}| \geq 4$  then  $L_i^{i+1}(\frac{3\pi}{4}) \subset D_i^{i+1} \subset L_i^{i+1}(\pi)$ .*

*Sketch of proof.* We first observe that  $L_i^{i+1}(\alpha)$  is symmetric with respect to the lines  $y = x$  and  $y = 2\nu_{i+1}^i + 2\pi - x$ , so it suffices to prove the inclusions in one quadrant. We focus on the quadrant  $\nu_{i+1}^i \leq x \leq y \leq 2\nu_{i+1}^i + 2\pi - x$ , and show that in this quadrant, the boundary of  $L_i^{i+1}(\alpha)$  consists of points whose corresponding shortest path has type  $L_\alpha SL$  and  $L_\alpha SR$ . We then obtain analytical expressions for these two arcs and prove that the segment bounding  $D_i^{i+1}$  in that quadrant separates the arcs for  $\alpha = \frac{3\pi}{4}$  and  $\alpha = \pi$ .  $\square$

We then extend this construction to an arbitrary number of points by defining the *diamond* of  $p_1, \dots, p_n$  as

$$\mathcal{D} = \bigcap_{i=1}^{n-1} (\mathbb{S}^1)^{i-1} \times D_i^{i+1} \times (\mathbb{S}^1)^{n-i-1}, \quad (3)$$

where  $D_i^{i+1}$  denotes the 2-dimensional diamond of  $p_i$  and  $p_{i+1}$  defined above (with the convention that  $(\mathbb{S}^1)^0 \times A = A \times (\mathbb{S}^1)^0 = A$ ). Notice the similarity with Equation (1):

$$\mathcal{L}(\alpha) = \bigcap_{i=1}^n (\mathbb{S}^1)^{i-1} \times L_i^{i+1}(\alpha) \times (\mathbb{S}^1)^{n-i-1}.$$

The inclusions  $L_i^{i+1}(\frac{3\pi}{4}) \subset D_i^{i+1} \subset L_i^{i+1}(\pi)$  from Lemma 15 then yield:

**Corollary 16.** *If  $p_1, \dots, p_n$  satisfy the  $(D_4)$  condition then  $\mathcal{L}(\frac{3\pi}{4}) \subset \mathcal{D} \subset \mathcal{L}(\pi)$ .*

We now describe the image of  $\mathcal{D}$  through the lifting  $(\mathbb{S}^1)^n \rightarrow \Lambda$ , defined in Section 4 from the family of hyperplanes  $\mathcal{H}$ .

**Lemma 17.**  *$\mathcal{D}$  does not intersect the hyperplanes of  $\mathcal{H}$ , and the lifting  $(\mathbb{S}^1)^n \rightarrow \Lambda$  maps the intersection of  $\mathcal{D}$  with each cell of  $(\mathbb{S}^1)^n \setminus \mathcal{H}$  to a convex polyhedron defined by at most  $4(n-1)$  linear inequalities in  $\mathbb{R}^n$ , each involving 2 variables.*

*Proof.* By definition,  $D_i^{i+1}$  intersects none of the lines  $\{(\theta_i, \theta_{i+1}) \in (\mathbb{S}^1)^2 \mid \theta_i = \nu_{i+1}^i\}$  and  $\{(\theta_i, \theta_{i+1}) \in (\mathbb{S}^1)^2 \mid \theta_{i+1} = \nu_{i+1}^i\}$ , thus  $\mathcal{D}$  intersects none of the hyperplanes of  $\mathcal{H}$ .

If  $I_1$  and  $I_2$  are two intervals in  $\mathbb{R}$  whose interiors avoid  $\nu_{i+1}^i + 2\pi\mathbb{Z}$ , then the image of  $D_i^{i+1}$  through the (partial) lift  $\mathbb{S}^1 \times \mathbb{S}^1 \rightarrow I_1 \times I_2$  is connected, and thus convex by definition of  $D_i^{i+1}$ . Now, every cell of  $(\mathbb{S}^1)^n \setminus \mathcal{H}$  is mapped through the lift  $(\mathbb{S}^1)^n \rightarrow \Lambda$  to a box of the type  $\Xi = \prod_i \Lambda_i^{\varepsilon_i}$  where  $\varepsilon_i \in \{-, +\}$ , and the image of  $\mathcal{D}$  through the (partial) lift  $(\mathbb{S}^1)^n \rightarrow \Xi$  is convex. In other words, the lift  $(\mathbb{S}^1)^n \rightarrow \Lambda$  maps each connected component of  $\mathcal{D}$  to a convex polyhedron in  $\mathbb{R}^n$ .

Each diamond  $D_i^{i+1}$  is defined by 4 linear inequalities involving  $\theta_i$  and  $\theta_{i+1}$ . Every such inequality, defined over  $(\mathbb{S}^1)^2$ , can be lifted through  $(\mathbb{S}^1)^2 \rightarrow \Lambda_i^{\varepsilon_i} \times \Lambda_{i+1}^{\varepsilon_{i+1}}$  to at most *one* inequality in  $\mathbb{R}^2$ , since the image of  $D_i^{i+1}$  through  $(\mathbb{S}^1)^2 \rightarrow \Lambda_i^{\varepsilon_i} \times \Lambda_{i+1}^{\varepsilon_{i+1}}$  is convex. Hence, the image of  $\mathcal{D}$  through the (partial) lift  $(\mathbb{S}^1)^n \rightarrow \Xi$  is a convex polyhedron; this polyhedron can be defined by at most  $4(n-1)$  linear inequalities in  $\mathbb{R}^n$ , each involving 2 variables. The result follows.  $\square$

We can already prove that computing a global shortest path through a sequence of points reduces to solving  $O(2^n)$  convex optimizations problems in  $n$  variables and with  $O(n)$  constraints each.

**Proposition 18.** *If  $p_1, \dots, p_n$  satisfy the  $(D_4)$  condition, any global minimum of  $F$  is contained in the image of  $\mathcal{D}$  through the lifting  $(\mathbb{S}^1)^n \rightarrow \Lambda \subseteq \mathbb{R}^n$ , which consists of at most  $2^{n-2}$  convex polyhedra, each defined by at most  $4(n-1)$  linear inequalities in 2 variables each. Furthermore,  $F$  is strictly convex over each of these convex polyhedra.*

*Proof.* The global minimum of  $F$  is realized over  $\mathcal{D}$  as it is realized over  $\mathcal{L}(\frac{3\pi}{4})$  (by Lemma 11) and  $\mathcal{L}(\frac{3\pi}{4}) \subset \mathcal{D}$  (by Corollary 16). For each cell  $c$  in  $(\mathbb{S}^1)^n \setminus \mathcal{H}$ , the lifting  $(\mathbb{S}^1)^n \rightarrow \Lambda$  maps  $\mathcal{D} \cap c$  to a convex polyhedron in  $\mathbb{R}^n$  defined by  $O(n)$  linear inequalities (by Lemma 17). Moreover, the length function  $F$  is convex on each such polyhedron since it is convex on  $\mathcal{L}(\pi)$  (by Proposition 7) and  $\mathcal{D} \subset \mathcal{L}(\pi)$  (by Corollary 16). Since  $(\mathbb{S}^1)^n \setminus \mathcal{H}$  has at most  $2^{n-2}$  cells, the statement follows.  $\square$

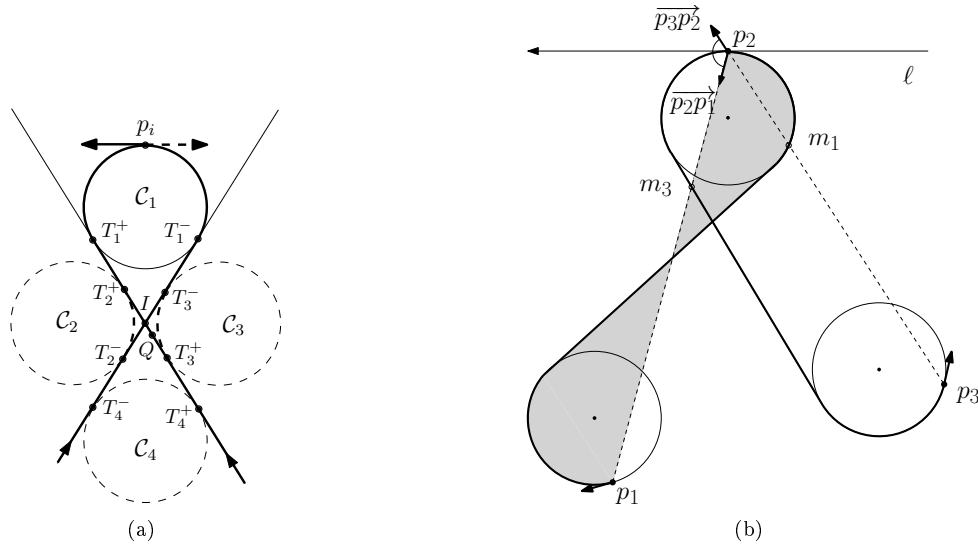


Figure 8: (a) Shortening a path that self-intersects. (b) If  $\theta_2 \in \Lambda_2^-$  and  $p_2$  is not a sharp turn, the path self-intersects. (The vectors are not drawn to scale for clarity.)

## 7 Pruning the connected components of $\mathcal{D}$

With Proposition 18, the computation of a global minimum of  $F$  reduces to solving a convex optimization problem on each cell of  $(\mathbb{S}^1)^n \setminus \mathcal{H}$ , where each of the cell is mapped to a product  $\prod_i \Lambda_i^{\varepsilon_i}$  through the lifting  $(\mathbb{S}^1)^n \rightarrow \Lambda$ ; in this section we give a condition on the respective positions of  $p_{i-1}$ ,  $p_i$  and  $p_{i+1}$  that implies that only the cells where  $\varepsilon_i = +$  can contain a global minimum of  $F$ . This will reduce the search from  $2^{n-2}$  to  $2^k$  cells, where  $k$  is the number of triples of consecutive points that violate our condition.

**Sharp turns and self-intersections.** Let  $\varphi_i$  denote the angle between  $\overrightarrow{p_i p_{i-1}}$  and  $\overrightarrow{p_i p_{i+1}}$  that is smaller or equal to  $\pi$ . We say that  $p_{i-1}, p_i, p_{i+1}$  form a *sharp turn*, or for simplicity that  $p_i$  is a *sharp turn*, if  $|\varphi_i| \leq \frac{\pi}{2}$  and if one of its neighbors,  $p_{i-1}$  and  $p_{i+1}$ , is within distance 4 from the segment formed by  $p_i$  and its other neighbor. Note that, if  $(p_{i-1}, p_i, p_{i+1})$  satisfies the  $(D_4)$  condition, then  $p_i$  is a sharp turn if and only if the latter condition is satisfied, that is,  $|\sin \varphi_i| \leq \frac{4}{\min(|p_{i-1} p_i|, |p_i p_{i+1}|)}$ .

We start by observing that “local” self-intersection of the globally shortest path through the via-points can be traced back to sharp turns.

**Lemma 19.** *If  $p_1, \dots, p_n$  satisfy the  $(D_4)$  condition and  $p_i$  is not a sharp turn then the portion, from  $p_{i-1}$  to  $p_{i+1}$ , of a globally shortest path has no self-intersection.*

*Proof.* Let  $\gamma$  denote the portion, from  $p_{i-1}$  to  $p_{i+1}$ , of a globally shortest path. We assume that  $\gamma$  self-intersects, and prove that  $p_i$  is a sharp turn. Let  $C_t^-$  and  $C_t^+$  denote the circular arcs of  $\gamma$  that precede and follow  $p_t$ , respectively, and  $S^-$  and  $S^+$  the line segments that precede and follow  $p_i$ . The self-intersection of  $\gamma$  is an intersection between two elements in  $\{C_{i-1}^+, S^-, C_i^-, C_i^+, S^+, C_{i+1}^-\}$ . We discuss the various situations in turn.

Assume first that two circular arcs intersect. It can be neither  $C_{i-1}^+$  and  $C_i^\pm$  (since  $|p_{i-1} p_i| > 4$ ), nor  $C_{i+1}^-$  and  $C_i^\pm$  (since  $|p_i p_{i+1}| > 4$ ), nor  $C_i^-$  and  $C_i^+$  (since their length is at most  $\frac{3\pi}{4}$  by Lemma 11). Then  $C_{i-1}^+$  intersects  $C_{i+1}^-$ , which implies that  $|p_{i-1} p_{i+1}| \leq 4$  and  $p_i$  is a sharp turn.

Any point in  $S^+$  is within distance at most 2 from the segment  $p_{i+1} p_i$ . Thus, if  $C_{i-1}^+$  intersects  $S^+$  the distance from  $p_{i-1}$  to the segment  $p_{i+1} p_i$  is at most 4, and  $p_i$  is a sharp turn. A similar argument handles the symmetric case where  $C_{i+1}^-$  intersects  $S^-$ . Since both  $S^+$  and  $S^-$  are tangent to the circle supporting  $C_i^-$  and  $C_i^+$  and to, respectively,  $C_{i+1}^-$  and  $C_{i-1}^+$ , no other intersection between a circular arc and a segment is possible.

Now assume that  $S^-$  and  $S^+$  intersect in some point  $I$ . Let  $L^-$  and  $L^+$  denote the two lines supporting  $S^-$  and  $S^+$ . We consider the four unit circles  $C_1, \dots, C_4$  tangent to both lines, and label the line/circle

intersections as shown in Figure 8(a). We assume that  $T_1^+, T_2^+, T_3^+$ , and  $T_4^+$  appear in this order on  $L^+$ ; this is without loss of generality because the arcs  $C_i^\pm$  are shorter than  $\frac{3\pi}{4}$ , by Lemma 11. If  $T_3^+$  lies on  $S^+$  and  $T_2^-$  lies on  $S^-$  then we can shorten  $\gamma$  using arcs of  $C_2$  and  $C_3$  (see Figure 8(a)), which contradicts the assumption that  $\gamma$  is part of a global shortest path. So, assume that  $T_3^+ \notin S^+$  (the other case is handled similarly) and let  $Q$  denote the endpoint of  $S^+$  other than  $T_1^+$ . Since  $Q$  lies between  $T_3^+$  and  $I$ , each of the two unit circles tangent to  $L^+$  at  $Q$  intersect  $L^-$ . The ray starting at  $T_1^-$  and containing  $S^-$  then intersects each of the disks of radius 2 centered in  $p_{i-1}$  and  $p_{i+1}$ . If that ray meets the disk centered in  $p_{i-1}$  first, then  $p_{i-1}$  is distance at most 4 from the segment  $p_i p_{i+1}$ , and a similar argument holds in the symmetric case. Hence,  $p_i$  is a sharp turn, which concludes the proof.  $\square$

**Sharp turns and tangents at  $p_i$ .** Let  $1 < i < n$ ,  $(\varepsilon_1, \dots, \varepsilon_n) \in \{-, +\}^n$  and consider a point  $(\theta_1, \dots, \theta_n) \in \prod_i \Lambda_i^{\varepsilon_i}$ . Let  $\gamma$  denote a shortest curvature-constrained path visiting the configurations  $(p_i, \theta_i)$  in order. Then  $\varepsilon_i = -$  if and only if the tangent to the path at  $p_i$  lies in the positive cone of the vectors  $\overrightarrow{p_i p_{i-1}}$  and  $\overrightarrow{p_i p_{i+1}}$  (see Figure 8(b)). We now show that if  $p_i$  is not a sharp turn then the tangent at  $p_i$  to any globally shortest path must lie outside this positive cone, and thus only the cells  $\prod_i \Lambda_i^{\varepsilon_i}$  with  $\varepsilon_i = +$  need to be considered when searching a global minimum of  $F$ .

**Lemma 20.** *If  $p_1, \dots, p_n$  satisfy the  $(D_4)$  condition, and  $p_i$  is not a sharp turn, then for any globally shortest path, the polar angle  $\theta_i$  of the tangent at  $p_i$  belongs to  $\Lambda_i^+$ .*

*Proof.* Let  $\gamma$  be a globally shortest path and assume, for a contradiction, that the polar angle  $\theta_i$  of its tangent vector at  $p_i$  is in  $\Lambda_i^-$ , and that  $p_i$  is not a sharp turn. We show that the arc of  $\gamma$  from  $p_{i-1}$  to  $p_{i+1}$  has a self-intersection, a contradiction with Lemma 19.

For simplicity, we consider  $i = 2$  and refer to Figure 8(b). Let  $\ell$  be the oriented line tangent to the (oriented) path  $\gamma$  at  $p_2$ , and, for any two distinct points  $a$  and  $b$ , let  $(ab)$  denote the oriented line from  $a$  to  $b$ . Without loss of generality, we assume that  $p_1$  is to the left<sup>11</sup> of  $\ell$ ; since  $\theta_i \in \Lambda_i^-$ ,  $p_3$  is to the left of  $\ell$  and to the right of  $(p_1 p_2)$ . Moreover, if  $p_3$  is on  $(p_1 p_2)$  then  $p_2$  is a sharp turn, so  $p_3$  is *strictly* to the right of  $(p_1 p_2)$  and, similarly,  $p_1$  is strictly to the left of  $(p_3 p_2)$ .

For  $j = 1, 3$  let  $\gamma_j$  denote the portion of  $\gamma$  between  $p_j$  and  $p_2$ . By Lemma 11, the circular arcs of  $\gamma_j$  have length at most  $\frac{3\pi}{4}$ , which is strictly less than  $\pi$ . Thus,  $p_1$  and  $p_3$  are strictly to the left of  $\ell$ , and  $\gamma_1$  and  $\gamma_3$  are also entirely strictly to the left of  $\ell$ , except for  $p_2$ .

We now argue that  $\gamma_1$  intersects the line  $(p_2 p_3)$  in  $p_2$  and exactly one other point, denoted  $m_1$ , at which  $\gamma_1$  traverses  $(p_2 p_3)$ . Let  $c$  be the number of intersection points between  $\gamma_1 \setminus p_2$  and  $(p_3 p_2)$ , counted with multiplicity. We first observe that  $\gamma$  crosses the line  $(p_3 p_2)$  from right to left in  $p_2$ , and  $p_1$  is strictly to the left of  $(p_3 p_2)$ , so  $c$  must be odd. Next,  $\gamma_1$  intersects any line other than the line supporting “its” segment, in at most three points, counted with multiplicity. Indeed, since the circular arcs have length at most  $\pi$ , if a circular arc meets the line in two points (possibly identical), then the segment does not intersect the line in another point, and the second circular arc intersects the line in at most one point (counted with multiplicity). Since  $p_2$  contributes at least one to this count,  $c$  must be at most two. Since  $c$  must also be odd,  $c = 1$  and  $\gamma_1$  intersects  $(p_2 p_3)$  in  $p_2$  and exactly one other point  $m_1$ , at which it traverses this line.

We furthermore prove that  $m_1$  belongs to the open segment  $[p_2 p_3]$ . First, since  $\gamma_1$  is to the left of  $\ell$ , so is  $m_1$ . As  $p_2$  is on  $\ell$  and  $p_3$  is strictly to the left of  $\ell$ , it follows that either  $m_1$  belongs to the segment  $[p_2 p_3]$ , or  $p_3$  belongs to the segment  $[p_2 m_1]$ . In the latter case,  $p_3$  lies in the convex hull of  $\gamma_1$  and is thus within distance at most 2 from the segment of  $\gamma_1$ ; this is impossible, as it would imply that  $p_3$  is at distance at most 4 from the segment  $p_1 p_2$ , i.e. that  $p_2$  is a sharp turn. Hence  $m_1$  belongs to the closed segment  $[p_2 p_3]$ . Finally,  $m_1 \neq p_2$  by definition, and  $m_1 \neq p_3$  because otherwise  $p_3$  lies in the convex hull of  $\gamma_1$ , again requiring that  $p_2$  be a sharp turn.

Similarly,  $\gamma_3$  intersects  $(p_1 p_2)$  in  $p_2$  and exactly one other point, denoted  $m_3$  which belongs to the open segment  $[p_1 p_2]$ .

Consider now the curve  $\rho$  obtained as the union of  $\gamma_1$  and the segment  $[p_1 p_2]$ . It is closed, and thus delimits a bounded region  $\mathcal{R}$  (not necessarily connected, see Figure 8(b)). Note that the line  $(p_2 p_3)$  meets  $\rho$  in exactly  $\{m_1, p_2\}$ , and  $m_1$  lies strictly in between  $p_2$  and  $p_3$ , thus  $p_3$  lies strictly outside the region  $\mathcal{R}$ .

<sup>11</sup>Unless specified otherwise, the constraint to be to the left (or to the right) of an oriented line is considered non-strict.

Consider finally the intersection between  $\gamma_3$  and  $\rho$ . Let  $\gamma'_j$  be  $\gamma_j$  minus its endpoint  $p_2$ .  $\gamma'_3$  intersects the line  $(p_1p_2)$  in exactly one point,  $m_3$ , at which it traverses  $(p_2p_3)$ . Since  $m_3$  lies on the open segment  $[p_1p_2]$ , either  $m_3$  is a (the) point of self-intersection of  $\rho$ , or  $\gamma'_3$  intersects the interior of  $\mathcal{R}$  in a neighborhood of  $m_3$ . In the former case,  $m_3$  then lies on  $\gamma'_1$  and thus  $\gamma$  is self-intersecting between  $p_1$  and  $p_3$ , contradicting Lemma 19. In the latter case, when  $\gamma'_3$  intersects the interior of  $\mathcal{R}$ ,  $\gamma'_3$  must intersect  $\rho$  in some other point because  $\gamma'_3$  does not intersect  $\mathcal{R}$  in some neighborhoods of  $p_2$  and  $p_3$ . Since  $\gamma_3$  is simple and intersects line  $(p_2p_3)$  only at  $p_2$  and  $m_3$ ,  $\gamma'_3$  must intersect  $\gamma'_1$ . Then, again,  $\gamma$  is self-intersecting between  $p_1$  and  $p_3$ , concluding the proof.  $\square$

**Proof of Theorem 1.** We can now complete the proof of our main result (Theorem 1) which states the following:

*Let  $p_1, \dots, p_n$  be a sequence of points in the plane that satisfy the  $(D_4)$  condition and has  $k$  sharp turns. All global minima of  $F$  are realized in a domain of  $(\mathbb{S}^1)^n$  that can be lifted to a union of up to  $2^k$  disjoint convex polyhedra, each defined by  $O(n)$  linear inequalities in  $\mathbb{R}^n$ . Moreover, through this lifting,  $F$  is strictly convex over each of these polyhedra.*

*Proof of Theorem 1.* By Proposition 18, under the  $(D_4)$  condition, the global minima of the length function  $F$  are contained, through the lifting, in  $2^{n-2}$  disjoint convex polyhedra, each defined by  $O(n)$  linear inequalities, over which  $F$  is strictly convex. If  $p_i$  is a non-sharp turn then any global minimum of  $F$  has its  $\theta_i$ -coordinate in  $\Lambda_i^+$ . The number of cells of  $(\mathbb{S}^1)^n \setminus \mathcal{H}$  that remains to explore is thus  $2^k$ , and for each of them it suffices to solve a convex optimization problem on a  $n$ -dimensional polyhedron defined by  $O(n)$  linear inequalities.  $\square$

## 8 Convex optimization

There are essentially two general methods that solve convex optimization problems with guaranteed complexity, the ellipsoid method and the class of interior point methods. While the latter is usually more efficient, it only works if one can compute so-called *self-concordant barriers* for the function to be minimized and its constraints. In our problem, the function  $F$  to be minimized is defined as the minimum of four functions which use inverse trigonometric functions, and computing *self-concordant barriers* currently seems out of reach. We thus focus on the ellipsoid method. We consider an extended real RAM model where arithmetic operations, trigonometric and inverse trigonometric functions, and the min function can be evaluated in constant time over the reals.

**Overview of the ellipsoid method.** Let us recall briefly its principles as described in the book of Ben-Tal and Nemirovski [4, Lecture 5] (see also the book of Gärtner and Matoušek [22, Chapter 7] for a quick overview). Given  $\varepsilon > 0$ , a convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , and a closed non-empty convex domain  $X \subset \mathbb{R}^n$ , we want to compute  $\hat{x} \in X$  such that  $f(\hat{x}) \leq \min_{x \in X} f(x) + \varepsilon$ . Here,  $X$  is one of the polyhedra over which we want to minimize  $F$ , and  $f$  is a convex extension over  $\mathbb{R}^n$  of the restriction of  $F$  to  $X$  (see Section 8.1 for details). The method starts with some ellipsoid  $\mathcal{E}_0$  (typically a ball) containing  $X$ , and constructs a sequence of ellipsoids  $(\mathcal{E}_i)_{i \geq 1}$  such that  $\mathcal{E}_i$  contains the minimum of  $f$  and the volume of  $\mathcal{E}_i$  decreases exponentially fast. Given the ellipsoid  $\mathcal{E}_i$  with center  $c_i$ ,  $\mathcal{E}_{i+1}$  is constructed as follows. A *separation oracle* is first used to test if  $c_i$  is contained in the domain  $X$  and, if not, to obtain a hyperplane  $H$  separating it from the domain. If  $c_i$  belongs to the domain, a *first-order oracle* is used to compute  $f(c_i)$  and the gradient  $\vec{n}$  of  $f$  in  $c_i$ ; let  $H$  denote the hyperplane through  $c_i$  with normal  $\vec{n}$ . Then,  $\mathcal{E}_{i+1}$  is defined as the ellipsoid with minimum volume that contains the portion of  $\mathcal{E}_i$  bounded by  $H$  and containing the minimum of  $f$ . At step  $i$ , an approximation  $\hat{x}$  is given by the center  $c_j$  of  $\mathcal{E}_j$ ,  $j \leq i$ , that lies in  $X$  and minimizes  $f$ . The cost of one iteration of this construction is quadratic in the number of constraints.

**Lemma 21.**  $\mathcal{E}_{i+1}$  can be computed from  $\mathcal{E}_i$  in  $O(n^2)$  time.

*Proof.* Testing if  $c_i$  is contained in the domain  $X$  takes  $O(n)$  time, since the domain is defined by  $O(n)$  linear inequalities, each involving only two variables (by Lemma 17). If  $c_i$  violates one inequality, we can obtain a separating hyperplane  $H$  from that inequality in  $O(1)$  time. If  $c_i$  is in the domain, we can



compute  $f(c_i) = F(c_i)$  by adding up the length of the shortest CSC path joining each pair of consecutive configurations, and the gradient of  $F$  in  $c_i$  by using Proposition 3; altogether this takes  $O(n)$  time. The complexity of the computation of  $\mathcal{E}_{i+1}$  from  $\mathcal{E}_i$  and  $H$  is dominated by the multiplication of a  $n \times n$  matrix and a  $n$ -dimensional vector [4, Lecture 5]; it can thus be done in  $O(n^2)$  time.  $\square$

**Number of iterations.** The number of steps needed to achieve an additive error of  $\varepsilon$  on the solution depends on the geometry of the domain:

**Theorem 22** ([4, Theorem 5.2.1]). *Let the convex set  $X$  of the problem contain a Euclidean ball of a given radius  $r > 0$  and be contained in the ball  $\mathcal{E}_0 = \{\|x\|^2 < R\}$  of a given radius  $R$ . For every input accuracy  $\varepsilon > 0$ , the ellipsoid method terminates after no more than*

$$N(\varepsilon) = \text{Ceil}\left(2n^2\left(\ln\left(\frac{R}{r}\right) + \ln\left(\frac{\varepsilon + \text{Var}_R(f)}{\varepsilon}\right)\right)\right) + 1$$

steps, where

$$\text{Var}_R(f) = \max_{x \in \mathcal{E}_0} f(x) - \min_{x \in \mathcal{E}_0} f(x).$$

Moreover the result  $\hat{x}$  generated by the method belongs to  $X$  and satisfies  $f(\hat{x}) \leq \min_{x \in X} f(x) + \varepsilon$ .

In our setting, bounding  $\frac{R}{r}$  and  $\text{Var}_R(f)$  requires an analysis of the geometry of the domain. If we consider any cell in  $(\mathbb{S}^1)^n \setminus \mathcal{H}$  then already for  $n = 3$  the ratio  $\frac{R}{r}$  can be arbitrarily large<sup>12</sup>. There is, however, one cell for which we can bound  $N(\varepsilon)$  in terms of  $n$  and  $\varepsilon$  (we postpone the rather technical proof to Section 8.1):

**Lemma 23.** *For any  $\varsigma > 2 + \sqrt{5}$ , if condition  $(D_\varsigma)$  holds and the considered domain  $X$  lies in  $\prod_{1 \leq i \leq n} \Lambda_i^+$ , then  $N(\varepsilon) = O(n^2 \ln \frac{n}{\varepsilon})$ .*

Recall that, if none of the  $p_i$  is a sharp turn, then, by Lemmas 17 and 20, the minimum of  $F$  lies in a convex polyhedron contained in  $\prod_{1 \leq i \leq n} \Lambda_i^+$ . Thus, this cell is the only one we need to consider, and the convexity of  $F$  ensures that there is a unique globally shortest path. Moreover, we have:

**Corollary 24.** *Let  $\varsigma > 2 + \sqrt{5}$ . If  $p_1, \dots, p_n$  is a sequence of points in the plane that satisfy the  $(D_\varsigma)$  condition and has no sharp turn then we can compute in time  $O(n^4 \log \frac{n}{\varepsilon})$  a path of curvature at most 1 that visits the  $p_i$  in order and whose length is at most that of the globally shortest path plus  $\varepsilon$ .*

## 8.1 Proof of Lemma 23

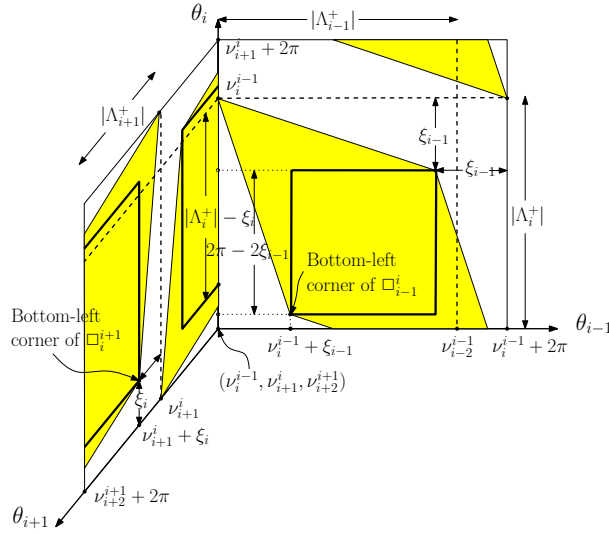
We give bounds on  $r$ ,  $R$  and  $\text{Var}_R(f)$ , and the statement will follow from the expression of  $N(\varepsilon)$  from Theorem 22. Bounding  $R$  is easy, as the domain is contained in  $\Lambda$  which is, by construction, contained in  $[-2\pi, 2\pi]^n$ ; we then have that  $R^2 \leq n(2\pi)^2$  and  $R = O(\sqrt{n})$ .

Bounding  $r$  is less straightforward. Recall that the diamond  $D_i^{i+1}$  is defined as the image of the quadrilateral with vertices  $(0, 2\pi)$ ,  $(\xi_i, \xi_i)$ ,  $(2\pi, 0)$ , and  $(2\pi - \xi_i, 2\pi - \xi_i)$  under the translation of vector  $(\nu_{i+1}^i, \nu_{i+1}^i)$  where  $\xi_i = \frac{2\pi}{d_i - \frac{1}{d_i}}$  (see Section 6). Let  $\square_i^{i+1}$  be the square in  $\Lambda_i \times \Lambda_{i+1}$  of side-length  $2\pi - 2\xi_i$  whose bottom-left corner is  $(\nu_{i+1}^i + \xi_i, \nu_{i+1}^i + \xi_i)$ ; see Figure 9. Just like we proceeded to define  $\mathcal{D}$  from the  $D_i^{i+1}$ , we extend each  $\square_i^{i+1}$  into a cylinder and intersect them all:

$$\square = \bigcap_{i=1}^n \left( \prod_{1 \leq j < i} \Lambda_j \right) \times \square_i^{i+1} \times \left( \prod_{i+1 < j \leq n} \Lambda_j \right),$$

with the convention that  $A \times \prod_{j \in J} \Lambda_j = A$  whenever  $J = \emptyset$ . Since  $\square_i^{i+1}$  is contained in the lift of  $D_i^{i+1}$ , it follows that  $\square$  is contained in the lift of  $\mathcal{D}$ . Let  $\Lambda^+ = \prod_{1 \leq i \leq n} \Lambda_i^+$ . Now, any ball contained in the box  $\square \cap \Lambda^+$  is contained in our domain. It thus remains to bound the side-length of  $\square \cap \Lambda^+$  from below to get a lower bound on  $r$ .

<sup>12</sup>Consider three points at  $p_1 = (-d, 0)$ ,  $p_2 = (0, 2)$  and  $p_3 = (d, 0)$ , and consider the cell  $\Lambda_1 \times \Lambda_2^- \times \Lambda_3$  (see Section 4) as  $d$  goes to infinity. Since the length of  $\Lambda_1$  and  $\Lambda_3$  is equal to  $2\pi$ , independently of  $d$ ,  $R$  remains larger than  $\pi$ . However, since the length of  $\Lambda_2^-$  goes to 0, so does  $r$  and the ratio  $R/r$  is therefore unbounded.

Figure 9: Determining a non-empty hypercube in the cell  $X \subseteq \prod_{1 \leq i \leq n} \Lambda_i^+$ .

The projection of  $\square \cap \Lambda^+$  on the  $i^{\text{th}}$ -coordinate axis is  $A \cap B$ , where  $A$  and  $B$  denote, respectively, the projections of  $\square_{i-1}^i \cap (\Lambda_{i-1}^+ \times \Lambda_i^+)$  and  $\square_i^{i+1} \cap (\Lambda_i^+ \times \Lambda_{i+1}^+)$  on the  $i^{\text{th}}$ -coordinate axis. The length of  $A$  is either  $2\pi - 2\xi_{i-1}$  or  $|\Lambda_i^+| - \xi_{i-1}$  and that of  $B$  is either  $2\pi - 2\xi_i$  or  $|\Lambda_i^+| - \xi_i$  (see Figure 9). Let  $\xi = \max_i \xi_i$ . Since  $|A| + |B| - |A \cap B| \leq |\Lambda_i^+|$ , we are in one of the three cases:

$$|A \cap B| \geq 4\pi - 4\xi - |\Lambda_i^+|, \quad \text{or} \quad |A \cap B| \geq 2\pi - 3\xi, \quad \text{or} \quad |A \cap B| \geq |\Lambda_i^+| - 2\xi.$$

Since  $2\pi \geq |\Lambda_i^+| \geq \pi$  we get in all three cases that

$$|A \cap B| \geq \min(2\pi - 4\xi, 2\pi - 3\xi, \pi - 2\xi) = \pi - 2\xi \quad \text{if } \xi \leq \frac{\pi}{2}.$$

By solving a degree-two equation, we get that  $\xi_i = \frac{2\pi}{d_i - \frac{1}{d_i}} < \frac{\pi}{2}$  is equivalent to  $d_i > 2 + \sqrt{5}$ . Thus, for any constant  $\varsigma > 2 + \sqrt{5}$  the  $(D_\varsigma)$  condition ensures that  $\xi < \frac{\pi}{2} - \varsigma'$  for some positive constant  $\varsigma'$ , and thus that  $|A \cap B|$  and the side-length of  $\square \cap \Lambda^+$  is bounded from below by some positive constant.<sup>13</sup> This box contains a ball of radius  $\Omega(1)$ , and thus  $r = \Omega(1)$ .

We now turn our attention to  $\text{Var}_R(f)$ . We start by showing how  $F$  can easily be extended into a convex function,  $f$ , over  $\mathbb{R}^n$ . Let  $H_i^{i+1}$  denote the family of all the tangent planes to the graph of  $F_i^{i+1}$  over the intersection of  $\Lambda_i^+ \times \Lambda_{i+1}^+$  with the lift of  $D_i^{i+1}$  to  $\Lambda_i \times \Lambda_{i+1}$ . We let  $f_i^{i+1}(x_i, x_{i+1})$  denote the function over  $\mathbb{R}^2$  whose graph is the lower envelope of  $H_i^{i+1}$ , let  $f = \sum_{i=1}^{n-1} f_i^{i+1}$ , and note that  $f$  is convex over  $\mathbb{R}^n$  and coincides with  $F$  over the domain.

We now bound  $\text{Var}_R(f)$ . First, we have that  $F_i^{i+1}(\theta_i, \theta_{i+1}) \geq |p_i p_{i+1}|$  and since the domain contains the global minimum of  $F$  in its interior,  $H_i^{i+1}$  contains a horizontal hyperplane and thus the same lower bound holds for  $f_i^{i+1}(\theta_i, \theta_{i+1})$ . It follows that:

$$\min_{x \in \mathcal{E}_0} f(x) \geq \sum_{i=1}^{n-1} |p_i p_{i+1}|.$$

Next, since the slope of any plane in  $H_i^{i+1}$  is at most 2 by Proposition 3, we have that

$$\max_{x \in \mathcal{E}_0} f(x) \leq \sum_{i=1}^{n-1} (|p_i p_{i+1}| + O(1)) \leq \min_{x \in \mathcal{E}_0} f(x) + O(n),$$

and  $\text{Var}_R(f) = O(n)$ .

Now, injecting  $R = O(\sqrt{n})$ ,  $r = \Omega(1)$  and  $\text{Var}_R(f) = O(n)$  into the statement of Theorem 22, we obtain that  $N(\varepsilon) = O(n^2 (\log \sqrt{n} + \log \frac{n}{\varepsilon})) = O(n^2 \log \frac{n}{\varepsilon})$ .

<sup>13</sup>Note that if  $(D_{2+\sqrt{5}})$  is violated, then  $\xi_i$  may be larger than  $\frac{\pi}{2}$  and  $A \cap B$ , and therefore  $\square \cap \Lambda^+$ , may be empty.

## 9 Extensions

In many situations, the initial and/or the final directions of the path are fixed and given. We first show, in Section 9.1, how our results can easily be extended to this variant of the problem. Next, we discuss the connections between the problems of finding curvature-constrained shortest paths (i) in the presence of polygonal obstacles, and (ii) through a given sequence of points, without obstacles.

### 9.1 Fixed initial and/or final directions

Our approach essentially relies on four properties (holding under various distance assumptions): (i)  $F$  is locally strictly convex over  $\mathcal{L}(\pi)$ , (ii) any global minimum belongs to  $\mathcal{L}(\frac{3\pi}{4})$ , (iii) there exists a shape  $\mathcal{D}$  sandwiched in-between  $\mathcal{L}(\frac{3\pi}{4})$  and  $\mathcal{L}(\pi)$  whose connected components are lifted to convex polyhedra, and (iv) every non-sharp turn reduces the number of connected components of  $\mathcal{D}$  to consider. We now explain how these properties can be extended when the initial and final directions (i.e., the values of  $\theta_1$  and  $\theta_n$ ) are fixed.

**Property (i).** The proof of Proposition 4 (local convexity of  $F_T(\theta_1, \theta_2)$ ) immediately yields that, when  $\theta_1$  is fixed, for any path type  $T$ , the map  $\theta_2 \mapsto F_T(\theta_1, \theta_2)$  is locally strictly convex whenever the path exists, the length of the circular arc at  $p_2$  is in  $(0, \pi)$ , and the other circular arc does not vanish. Note that, the length of the circular arc at  $p_1$  may be larger than  $\pi$  and thus Proposition 5 (uniqueness of the shortest *CSC*-path when both arcs are shorter than  $\pi$ ) cannot be used directly. It follows that it is unclear whether Theorem 6 can be extended to showing that  $\theta_2 \mapsto \min_{T \in \{LSL, LSR, RSL, RSR\}} F_T(\theta_1, T_2)$  is locally strictly convex. Nonetheless, we easily get that each of

$$\theta_2 \mapsto \min_{T \in \{LSL, LSR\}} F_T(\theta_1, \theta_2) \quad \text{and} \quad \theta_2 \mapsto \min_{T \in \{RSR, RSL\}} F_T(\theta_1, \theta_2) \quad (4)$$

is locally strictly convex whenever the length of the circular arc at  $p_2$  is less than  $\pi$ , and the other circular arc does not vanish.<sup>14</sup>

Since the minimum of  $F_T$  over all possible  $T$  is not known to be locally convex, we consider  $F_\sigma(\theta_2, \dots, \theta_{n-1})$ ,  $\sigma = 1, \dots, 4$ , the length function of a shortest curvature-constrained path that goes through the configurations  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  in order, and such that the circular arcs incident to  $p_1$  and  $p_n$  have a fixed orientation  $L$  or  $R$  determined by  $\sigma$ ;  $\sigma = 1, \dots, 4$  corresponds to each of the four choices of orientations. Let  $\mathcal{L}_\sigma(\alpha)$  denote the set of  $(\theta_2, \dots, \theta_{n-1})$  such that the shortest path visiting  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  in order and whose type in  $p_1$  and  $p_n$  is prescribed by  $\sigma$  has all its circular arcs of length less than  $\alpha$ , except possibly the first and the last arc. The local convexity of the functions in (4), as well as the similar functions when  $\theta_2$  is fixed, yields, with the proof of Proposition 7 ( $F$  is locally convex over  $\mathcal{L}(\pi)$ ), the following analogue to Property (i):

**Proposition 25.** *For any fixed  $\theta_1$ ,  $\theta_n$ , and  $1 \leq \sigma \leq 4$ ,  $F_\sigma$  is locally strictly convex over  $\mathcal{L}_\sigma(\pi)$ .*

**Property (ii).** The necessary conditions (ii) and (iii) on locally shortest paths of Proposition 9 and the proof of Lemma 11 ( $\mathcal{L}(\pi)$  contains the global minima) rely on finding (local or global) shortcuts that would contradict the optimality of the path. These arguments can still be used in our setting as any locally/global shortest path that starts and ends with prescribed types of circular arcs is indeed a local/global minimum of  $F_\sigma$  for the corresponding  $\sigma$ . Since these shortcuts never consider the circular arcs incident to  $p_1$  or  $p_n$ , our property (ii) extends immediately:

**Lemma 26.** *For any fixed  $\theta_1$ ,  $\theta_n$ , and  $1 \leq \sigma \leq 4$ , any global minimum of  $F_\sigma$  belongs to  $\mathcal{L}_\sigma(\frac{3\pi}{4})$ .*

<sup>14</sup>It is sufficient to show that the minimum of the two functions  $F_{LSL}$  and  $F_{LSR}$  is  $C^1$  at any point where their graphs intersect (and similarly for  $F_{RSL}$  and  $F_{RSR}$ ). If two paths of types *LSL* and *LSR* have their second circular arc shorter than  $\pi$ , then exactly one of these two paths has its first circular arc shorter than  $\pi$  (indeed, by Proposition 5, both paths cannot have both arcs shorter than  $\pi$ , nor can they have both arcs longer than  $\pi$ , since otherwise both paths can be shortened by half a circle, yielding that two paths with both arcs are shorter than  $\pi$ ). Thus  $F_{LSL}$  and  $F_{LSR}$  take distinct values at any point where the length of the second circular arc of their corresponding paths is in  $(0, \pi)$ . When the second circular arc vanishes, the paths of type *LSL* and *LSR* coincide, and the minimum of  $F_{LSL}$  and  $F_{LSR}$  is smooth (as in the proof of Theorem 6).

**Property (iii).** For any  $1 < i < n - 1$ , the lemon  $L_i^{i+1}(\alpha)$  depends on neither  $\theta_1$  nor  $\theta_n$ , thus the diamond  $D_i^{i+1}$  defined in Section 6, still satisfies  $L_i^{i+1}(\frac{3\pi}{4}) \subset D_i^{i+1} \subset L_i^{i+1}(\pi)$ . For  $i = 1$  (and similarly for  $i = n - 1$ ), the lemon  $L_1^2(\pi)$  is one-dimensional since  $\theta_1$  is fixed and, moreover, it is a segment.<sup>15</sup>

The domain  $\mathcal{D}$  can then be adapted into a new domain  $\mathcal{D}_\sigma$  by simply setting  $D_1^2$  to be that segment (and, similarly, for  $D_{n-1}^n$ ). Lemma 17 then immediately extends:

**Lemma 27.** *For any fixed  $\theta_1$ ,  $\theta_n$ , and  $1 \leq \sigma \leq 4$ ,  $\mathcal{D}_\sigma$  does not intersect the hyperplanes of  $\mathcal{H}$ , and the lifting  $(\mathbb{S}^1)^n \rightarrow \Lambda$  maps the intersection of  $\mathcal{D}_\sigma$  with each cell of  $(\mathbb{S}^1)^n \setminus \mathcal{H}$  to a convex polyhedron defined by at most  $4(n - 1)$  linear inequalities in  $\mathbb{R}^n$ , each involving 2 variables.*

**Property (iv).** Finally, the proof of Lemma 20 uses the fact that the circular arcs at  $p_{i-1}$  and  $p_{i+1}$  are shorter than  $\pi$  which may be an issue when analyzing  $p_2$  and  $p_{n-1}$ . It does, however, generalize immediately for  $p_3, \dots, p_{n-2}$  and allows to reduce the search to  $2^{k+2}$  connected components where  $k$  is the number of sharp turns in  $\{p_3, \dots, p_{n-2}\}$ . From there, the results of Section 8 extend immediately.

We finally obtain the following analogue of Theorem 1:

**Theorem 28.** *Let  $p_1, \dots, p_n$  be a sequence of points in the plane that satisfy the  $(D_4)$  condition and let  $k$  denote the number of sharp turns at  $p_i$ ,  $3 \leq i \leq n - 2$ . For any fixed  $\theta_1$  and  $\theta_n$ , all curvature-constrained shortest paths from  $(p_1, \theta_1)$  to  $(p_n, \theta_n)$  that visit  $p_1, \dots, p_n$  in order can be computed by minimizing four functions over  $2^{k+2}$  polyhedra in  $\mathbb{R}^n$  over which they are strictly convex, each polyhedron being defined by at most  $4n - 1$  linear inequalities in two variables each.*

## 9.2 Shortest paths among polygonal obstacles

Any shortest path of bounded curvature between two given points *in the presence of polygonal obstacles* is a concatenation of Dubins paths whose extremities are the starting point, the ending point, and some *contact points* on the boundary of the obstacles [14, 17]. We now discuss how our results imply that, under some conditions, once the sequence of contact points is known, “filling the dots” reduces to computing locally shortest curvature-constrained paths visiting certain sequences of contact points *in the absence of obstacles* (Proposition 29), and admits a convex optimization formulation similar to Theorem 1 (Theorem 31). This suggests that the real difficulty in curvature-constrained path planning among polygonal obstacles resides in the subproblem of finding the sequence of contact points, a question that we do not address here.

**Problem statement.** Let  $\rho$  be a bounded-curvature shortest path between two configurations in the presence of polygonal obstacles. We call a circular arc of  $\rho$  *anchored* if it touches the obstacles in at least two points. The question we consider is the following: we assume that we are given the sequence  $p_1, \dots, p_n$  of contact points of  $\rho$  with the obstacles as well as the anchored circular arcs contained in  $\rho$  and we want to reconstruct the whole geometry of the path. As in the previous sections, we work under the (restrictive) condition that  $\rho$  is a concatenation of *CSC*-paths with non-vanishing line segments; in other words, *we rule out shortest paths that contain two consecutive non-vanishing circular arcs*.

To keep the presentation simple, and without loss of generality, we assume that  $\rho$  touches the obstacles in a finite number of points (since the case where it travels along the boundary of an obstacle can be handled easily) and that we know the orientation of the circular arcs following the first point and preceding the last point of  $\rho$  (we discuss after Theorem 31 how to manage without this information).

The length-reducing perturbations of *SCS*-paths of Dubins [12, Lemma 1] imply that for any endpoint  $p_i$  of an anchored circular arc of  $\rho$ , the tangent to  $\rho$  in  $p_i$  and the orientation,  $L$  or  $R$ , of the circular arcs preceding and following  $p_i$  are prescribed by the geometry of the obstacle in  $p_i$ . Splitting  $\rho$  at the endpoints of circular arcs, we obtain a sequence of subpaths that contain no anchored circular arcs and join known initial and final configurations. We can thus assume, again without loss of generality, that  $\rho$  *contains no anchored circular arc*.

<sup>15</sup>If a path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  of type *LSL* or *LSR* uses an arc of length exactly  $\pi$  ending in  $p_2$  then it uses a segment that is tangent to both the  $L$  circle of  $(p_1, \theta_1)$  and the circle of radius 2 centered in  $p_2$ . Since there can be at most two such segments (because of orientation of circle  $L$ ), it follows that for any fixed  $\theta_1$  and  $\sigma$  the corresponding lemon  $L_1^2(\pi)$  has at most two endpoints and thus is a segment.

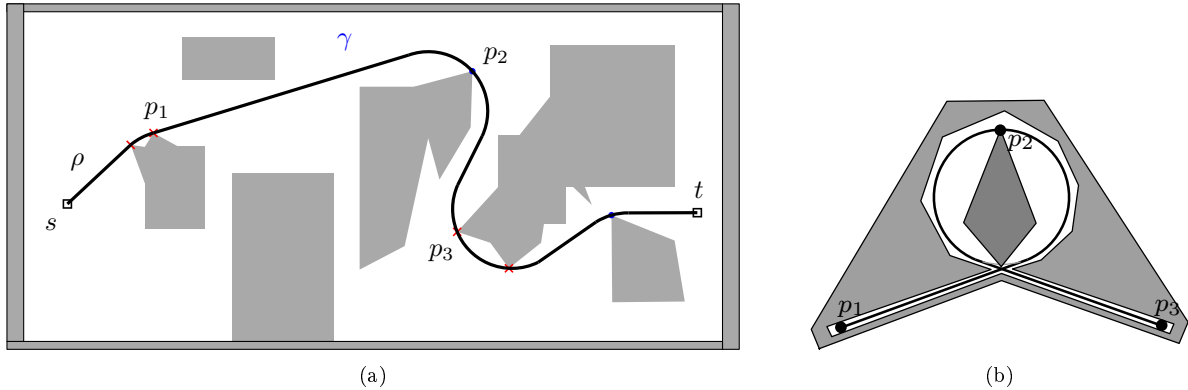


Figure 10: (a) A subpath  $\gamma$  of the shortest path  $\rho$  from  $s$  to  $t$ ;  $\gamma$  has one via-point,  $p_2$ . (b) A shortest path in the presence of obstacles may have, in general, circular arcs of length arbitrarily close to  $\pi$ , preceding and following a via-point.

**Local geometry of  $\rho$ .** Let  $\tilde{\theta}_i$  denote the polar angle of the tangent to  $\rho$  in  $p_i$ ; we are given  $\tilde{\theta}_1$  and  $\tilde{\theta}_n$  and our first step is to characterize  $\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1}$ . We use a separate set of “free” variables  $\theta_2, \dots, \theta_{n-1}$  and, for the sake of the presentation, we also consider  $\theta_1$  and  $\theta_n$  which are fixed equal to  $\tilde{\theta}_1$  and  $\tilde{\theta}_n$ , respectively. As in Section 9.1, we let  $F_\sigma(\theta_2, \dots, \theta_{n-1})$  denote the length of a shortest curvature-constrained path that goes, in the absence of obstacles, through the configurations  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  in order and such that the circular arcs incident to  $p_1$  and  $p_n$  have the prescribed orientation.<sup>16</sup> Note that here  $\sigma$  is fixed and corresponds to the prescribed orientation at  $p_1$  and  $p_n$  of the optimal path  $\rho$ . We can now state our first result.

**Proposition 29.**  *$(\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1})$  is a local minimum of  $F_\sigma$ . Moreover, for  $i = 2, \dots, n-2$ , the part of  $\rho$  between  $p_i$  and  $p_{i+1}$  coincide with the shortest CSC-path from  $(p_i, \tilde{\theta}_i)$  to  $(p_{i+1}, \tilde{\theta}_{i+1})$  in the absence of obstacles.*

We remark that for  $i = 1$  and  $i = n-1$ , the shortest CSC-path in the absence of obstacle may collide with the obstacles.

*Proof.* We can assume that  $n > 2$  since otherwise there is nothing to prove. Thus,  $\rho$  cannot be reduced to a single circular arc, and the obstacles touch every circular arc of  $\rho$  exactly once.

Let us first assume that  $\rho$  has no vanishing circular arc. Let  $T_i \in \{LSR, RSL, LSL, RSR\}$  be the type of the CSC-subpath of  $\rho$  from  $(p_i, \tilde{\theta}_i)$  to  $(p_{i+1}, \tilde{\theta}_{i+1})$ ,  $i = 1, \dots, n-1$ . We define  $F_T(\theta_1, \dots, \theta_n)$  as the length of the path through  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  whose type between the  $i$ -th and  $(i+1)$ -th configurations is  $T_i$ . We note that  $F_T(\theta_2, \dots, \theta_{n-1})$  measures the length of  $\rho$ .

The length-reducing perturbations of CSC-paths of Dubins [12, Lemma 1] imply that at each contact point  $p_2, \dots, p_{n-1}$  the obstacle is locally inside the disk supporting the circular arcs. Thus, each non-terminal circular arc of  $\rho$  can rotate slightly around the contact point it contains without creating any new intersection with the obstacles. Since the path the length of which is measured by  $F_T(\theta_2, \dots, \theta_{n-1})$  deforms continuously as  $(\theta_2, \dots, \theta_{n-1})$  changes, we have that in a neighborhood of  $(\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1})$  this path does not cross the obstacles properly. It follows that  $(\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1})$  is a local minimum of  $F_T$ .

The proof of Lemma 12 now implies that the two circular arcs preceding and following every point  $p_i$  ( $1 < i < n$ ) have the same orientation ( $R$  or  $L$ ), and their lengths are either equal or sum up to  $2\pi$ .<sup>17</sup> Since  $\rho$  is a globally shortest path, the total length of the two arcs incident to  $p_i$  is strictly less than  $2\pi$ .

<sup>16</sup>If the initial (resp. final) circular arc of  $\rho$  vanishes then  $F_\sigma$  makes no requirement on the type of the initial (resp. final) circular arc.

<sup>17</sup>The statement in Lemma 12 concerning the nonterminal points  $p_i$  ( $1 < i < n$ ) was proved by considering paths from  $(p_{i-1}, \tilde{\theta}_{i-1})$ , through  $(p_i, \theta_i)$ , and to  $(p_{i+1}, \tilde{\theta}_{i+1})$  (with  $\tilde{\theta}_{i-1}, \tilde{\theta}_{i+1}$  fixed); the fact that the direction of the path at  $p_1$  and  $p_n$  was not constrained in Lemma 12 has thus no impact here. Note also that, formally, the path  $\gamma$  we considered in the statement of Lemma 12 was, for ease of exposition, a *shortest* Dubins path between every two consecutive configurations. This is not the case here, since the Dubins path between any two consecutive configurations is not known to be the shortest. However, the proof of Lemma 12 does not use the hypothesis that the Dubins subpaths of  $\gamma$  are the *shortest*; it considers the type of the paths incident to each  $p_i$ , regardless of its optimality.

It follows that the arcs incident to every  $p_i$ ,  $1 < i < n$ , have the same orientation and the same length, which is less than  $\pi$ . Hence, by Proposition 5, the path  $\rho$  between every two nonterminal consecutive configurations is the shortest *CSC*-path. It follows that in a neighborhood of  $(\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1})$ ,  $F_\sigma$  and  $F_{\mathcal{T}}$  coincide. This proves both results.

If some circular arc of  $\rho$  vanishes then we relax the definition of  $F_{\mathcal{T}}$  by allowing the corresponding circular arc to have an arbitrary type. This ensures that the path the length of which is measured by  $F_{\mathcal{T}}$  still deforms continuously when  $(\theta_2, \dots, \theta_{n-1})$  changes so  $(\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1})$  is still a local minimum of  $F_{\mathcal{T}}$ . We thus get that the circular arcs preceding and following each contact point have equal length and are shorter than  $\pi$ ; from there, Proposition 5 implies both statements.  $\square$

We stress that Proposition 29 states that, in the presence of obstacles, a bounded-curvature shortest path that contains no *CC* subpath and no anchored circular arc is such that every nonterminal *CSC*-subpath connecting two configurations touching the obstacles is *the* shortest *CSC*-path in the *absence* of obstacles. We believe that this property of shortest paths among obstacles is new; in particular, it strengthens the result of Fortune and Wilfong [14], and Jacobs and Canny [17] that a shortest path in the presence of obstacles is a concatenation of *Dubins* paths.

**Reduction to convex optimization.** Proposition 29 suggests that the problem of computing a bounded-curvature shortest path in the presence of obstacles, given the sequence of its contact points with the obstacles and its initial and final orientation, can be broken down into computing minima of length functions  $F_\sigma$  and therefore may be amenable to the machinery we developed in Sections 2–7.

As described in Section 9.1, and with the same definition of  $\mathcal{L}_\sigma(\alpha)$ ,<sup>18</sup> our approach relies on four main properties : (i)  $F_\sigma$  is locally strictly convex over  $\mathcal{L}_\sigma(\pi)$ , (ii) any global minimum belongs to  $\mathcal{L}_\sigma(\frac{3\pi}{4})$ , (iii) there exists a shape  $\mathcal{D}_\sigma$  sandwiched in-between  $\mathcal{L}_\sigma(\frac{3\pi}{4})$  and  $\mathcal{L}_\sigma(\pi)$  whose connected components are lifted to convex polyhedra, and (iv) every non-sharp turn reduces the number of connected components of  $\mathcal{D}_\sigma$  to consider.

In our setting, Properties (i) and (iii) hold exactly as in Section 9.1. However, Property (ii) does not directly hold for several reasons. First, the minima we are seeking are only guaranteed to be *local minima* (Proposition 29), whereas our machinery was developed for *global minimum* (and the globality of the minimum was used for showing that it belongs to  $\mathcal{L}_\sigma(\frac{3\pi}{4})$ ). Second, the local minima we are seeking may actually not belong to  $\mathcal{L}_\sigma(\frac{3\pi}{4})$ : Figure 10(b) shows an example in which a bounded-curvature shortest path from  $p_1$  to  $p_3$  in the presence of obstacles is, in the absence of obstacles, a locally shortest path through  $p_1, p_2, p_3$  where the length of the circular arcs preceding and following  $p_2$  can be made arbitrarily close to  $\pi$ .

Such situations can, however, be circumvented by assumptions that are reasonable in many contexts. Say that the path  $\rho$  is *locally simple* in  $p_i$  if the portion of  $\rho$  from  $p_{i-1}$  to  $p_{i+1}$  has no self-intersection.

**Lemma 30.** *If  $p_1, \dots, p_n$  satisfy the  $(D_{2+\sqrt{5}})$  condition and if each  $p_i$  in which  $\rho$  is not locally simple is contained in an obstacle of diameter larger than  $1 + \sqrt{2}$ , then  $(\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1})$  belongs to  $\mathcal{L}_\sigma(\frac{3\pi}{4})$ .*

*Proof.* Assume, for a contradiction, that at least one of the circular arcs incident to some  $p_i$ ,  $1 < i < n$ , has length at least  $\frac{3\pi}{4}$ . By Proposition 9, since  $(\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1})$  is a local minimum of  $F_\sigma$ , both circular arcs at  $p_i$ ,  $1 < i < n$ , have same length or their lengths sum up to  $2\pi$ . The latter case contradicts the optimality of  $\rho$ . In the former case, condition  $(D_{2+\sqrt{5}})$  and Lemma 2 imply that the incident line segments are longer than 1 and the path therefore self-intersects. Moreover, the closed region formed by the two circular arcs and the two segments clipped at their intersection point is of diameter at most  $1 + \sqrt{2}$ . Thus, if the obstacles have larger diameters, they cannot be contained in that region. Hence, the concatenation of the two circular arcs at  $p_i$  can only touch the obstacles on its concave side, and it must touch the obstacles in at least two points (including  $p_i$ ) since  $\rho$  is a shortest path, contradicting the hypothesis that  $\rho$  contains no anchored circular arc. Therefore,  $(\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1})$  belongs to  $\mathcal{L}_\sigma(\frac{3\pi}{4})$ .  $\square$

We consider finally Property (iv). The path corresponding to a point in  $\mathcal{L}_\sigma(\frac{3\pi}{4})$  has all its circular arc shorter than  $\frac{3\pi}{4}$ , except possibly the arc incident to  $p_1$  and  $p_n$ . For such a path, the proof of Lemma 20 shows that, under the  $(D_4)$  condition (and thus also under  $(D_{2+\sqrt{5}})$ ), and for  $3 \leq i \leq n-2$ , if  $p_i$  is

<sup>18</sup>Recall, for clarity, that  $\mathcal{L}_\sigma(\alpha)$  denote the set of  $(\theta_2, \dots, \theta_{n-1})$  such that the shortest path (without obstacles) visiting  $(p_1, \theta_1), \dots, (p_n, \theta_n)$  in order satisfies two conditions: (i) the circular arcs following  $p_1$  and preceding  $p_n$  have their orientations prescribed by  $\sigma$ , that is the same as in  $\rho$ , and (ii) all the other circular arcs have length less than  $\alpha$ .

not a sharp turn and  $\theta_i$  is in  $\Lambda_i^-$ , then  $\rho$  is not locally simple in  $p_i$ ; thus if  $\rho$  is locally simple in  $p_i$ , and  $p_i$  is not a sharp turn, then  $\theta_i \in \Lambda_i^+$ . Putting everything together, we obtain the following analogue of Theorems 1 and 28.

**Theorem 31.** *Assume that  $p_1, \dots, p_n$  satisfy the  $(D_{2+\sqrt{5}})$  condition and that every  $p_i$  in which  $\rho$  is not known to be locally simple is contained in an obstacle of diameter larger than  $1 + \sqrt{2}$ . Let  $k$  be the number of  $p_i$ ,  $3 \leq i \leq n-2$ , that are sharp turns or in which  $\rho$  is not known to be locally simple.*

- (i)  *$(\tilde{\theta}_2, \dots, \tilde{\theta}_{n-1})$  is one of the local minima of  $F_\sigma$  over a collection of at most  $2^{k+2}$  convex polyhedra in  $\mathbb{R}^n$ , over which  $F_\sigma$  is strictly convex; every polyhedron is defined by at most  $4n-1$  linear inequalities in two variables each.*
- (ii) *For each of these at most  $2^{k+2}$  local minima  $(\theta_2, \dots, \theta_{n-1})$ , consider the concatenation of the shortest CSC-paths from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$ , for  $i = 1, \dots, n-1$  (with the circular arcs following  $p_1$  and preceding  $p_n$  of prescribed orientation). The path  $\rho$  is the (or a) shortest of these paths that avoids the obstacles.*

Note that, since  $\rho$  contains no anchored circular arc, if  $\rho$  is not locally simple at  $p_i$  then the loop of  $\rho$  passing through  $p_i$ , and joining the self-intersection to itself, must enclose an obstacle. In particular, if  $\rho$  is a bounded-curvature shortest path in a *simple polygon*, it is locally simple in every point, and  $k$  is the number of sharp turns among  $p_3, \dots, p_{n-2}$ .

**Unknown initial and final orientations.** If the orientation of the circular arcs of  $\rho$  following  $p_1$  and preceding  $p_n$  are unknown, a natural approach is to consider four distinct subproblems, one for each choice of orientations of these arcs. More precisely, we want to solve the problem (i) of finding a shortest path  $\rho$  between  $(p_1, \tilde{\theta}_1)$  to  $(p_n, \tilde{\theta}_n)$  and through  $p_2, \dots, p_{n-1}$  that avoids the obstacles. We have showed how to solve the problem (ii) where the orientation ( $L$  or  $R$ ) of  $\rho$  is *known* at  $p_1$  and  $p_n$ ; we now consider the problem (iii) of finding a shortest path  $\rho_\sigma$  where the orientation of  $\rho_\sigma$  is *prescribed* at  $p_1$  and  $p_n$ .

A difficulty with Problem (iii) is that if we enforce the path to turn, say, right at  $p_1$ , then it is possible that the circular arc of  $\rho_\sigma$  at  $p_1$  vanishes which implies that  $\theta_2$  is constrained; this situation can propagate and nothing ensures that the two circular arcs incident to a  $p_i$  have the same length. In other words the proof of Proposition 29 does not hold anymore. However, we note that this is not a problem in the end because the set of candidate shortest paths obtained by applying Theorem 31 to each length function  $F_\sigma$  is guaranteed to contain  $\rho$  as Proposition 29 is true for the choice of orientations that correspond to  $\rho$ .

## 10 Concluding remarks

We conclude with a few remarks on possible extensions of our results.

**Distance condition.** The requirement in Theorems 1 and 28 that the points  $p_1, \dots, p_n$  satisfy the  $(D_4)$  condition is used in three places. First, it excludes the occurrence of *CCC*-paths. Then, it is used to argue that the global minima of the length function  $F$  belong to the lemon  $\mathcal{L}(\frac{3\pi}{4})$  (Lemma 11). Finally, it is instrumental for proving that the minima of  $F$  can be searched for in convex components over which  $F$  is convex (Lemma 15). Further relaxing the distance condition in these theorems therefore seems a considerable task. In particular, this would require to study the convexity properties of the length function of *CCC* paths, task we did not undertake here since this paper is already substantial.

Corollary 24 uses a  $(D_\varsigma)$  condition, with  $\varsigma > 2 + \sqrt{5}$ , to bound the complexity of the ellipsoid method for computing the globally shortest path in the absence of sharp turn. While the reduction to a single convex optimization problem holds under the weaker distance condition  $(D_4)$ , we do not see how the complexity analysis could be extended for  $\varsigma \leq 2 + \sqrt{5}$ .

Last, Theorem 31, our extension of Theorem 1 in the presence of obstacles, considers the stronger distance assumption  $(D_{2+\sqrt{5}})$ . This assumption is only used to ensure, in Lemma 30, that the minima of the length function belong to  $\mathcal{L}(\frac{3\pi}{4})$ . This is done in a straightforward way and is likely to be improved, at least in specific settings such as inside a simple polygon.

**More efficient pruning.** The idea underlying Theorem 1 is that finding a bounded-curvature globally shortest path visiting a sequence of points in a prescribed order can be decomposed into two problems: (a) identifying which, among  $2^{n-2}$  convex polyhedra, contains a global minimum of the length function,<sup>19</sup> and (b) minimizing the length function over that polyhedron. We essentially solved problem (b) by showing that it admits a convex optimization formulation. Regarding problem (a), we observed that excluding certain self-intersections in the path reduces (in fact, halves) the set of candidate polyhedra; we encapsulated this condition in the coarser, but simpler and more intuitive, notion of sharp turns; exploring other conditions that reduce the combinatorial explosion of problem (a) is a natural follow-up question.

**Obstacles.** A natural question raised by our results of Section 9.2 is whether the sequence of points where the shortest path between two configurations in the presence of obstacles can be computed, or even approximated. A candidate setting in which this question could perhaps be tackled is inside a simple polygon, where the homotopy class of the shortest path is trivial.

## Acknowledgment

The authors would like to thank Jean-Daniel Boissonnat for his implication at a very early stage of this work, during the PhD of the third author, and Otfried Cheong for helpful discussions.

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<sup>19</sup>More technically, this amounts to deciding at each via-point  $p_i$ ,  $2 \leq i \leq n-1$ , whether the polar angle of the tangent to a globally shortest path at that point belongs to  $\Lambda_i^+$  or  $\Lambda_i^-$ .



## A Local convexity of the length function of a *CSC*-path

We present here complete proofs of the results claimed in Section 3. Recall that we prove here that the length  $F_{csc}(\theta_1, \theta_2)$  of a shortest *CSC*-path from configuration  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is a locally strictly convex function at any point  $(\theta_1, \theta_2)$  such that both circular arcs of the corresponding path are shorter than  $\pi$ . For simplicity, we refer in this section to  $F_{csc}$  as  $F$ . The angles  $\theta_1$  and  $\theta_2$  are considered, throughout this section, in  $\mathbb{R}$  rather than in  $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ . In other words, the function  $F$  is defined over  $\mathbb{R}^2$ , seen as the standard universal covering of the torus  $\mathbb{S}^1 \times \mathbb{S}^1$ . The (local) convexity of the function is thus defined over  $\mathbb{R}^2$  in the standard way.

We start by showing that for a given path type  $T \in \{LSR, RSL, LSL, RSR\}$ , the length  $F_T(\theta_1, \theta_2)$  of the  $T$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is such a locally strictly convex function (Proposition 35). We prove this by first computing the first derivatives and the Hessian of the length function (Propositions 33 and 34).

We then prove that the local convexity extends to the length  $F(\theta_1, \theta_2) = \min_{T \in \{LSR, RSL, LSL, RSR\}} F_T(\theta_1, \theta_2)$  of a shortest *CSC*-path (Theorem 37). We prove this by first showing the interesting property that, if both circular arcs of a *CSC*-path are shorter than  $\pi$ , then this path is *the* shortest *CSC*-path (Proposition 36).

**Notation.** We introduce here some new notation; we also recall, for clarity, the notation introduced in Section 3. Refer to Figure 11. For a given *CSC*-path, let  $O_i$ ,  $i = 1, 2$ , denote the center of the unit circle supporting the  $i$ -th circular arc (defined by continuity if one of the circular arcs vanish), let  $\alpha_i$  be the length of this  $i$ -th circular arc, and let  $M_1$  and  $M_2$  be the first and last endpoint of the line segment of the path. Let  $\vec{U}_{12}$  denote the vector  $\vec{M_1 M_2} / M_1 M_2$  where  $M_1 M_2$  denotes the Euclidean distance from  $M_1$  to  $M_2$ . Let  $\mu_B$  be equal to 1 if  $B$  is true and to  $-1$  otherwise. In particular, for a type of path  $T \in \{LSR, RSL, LSL, RSR\}$ ,  $\mu_{C_j=R}$  ( $j = 1, 2$ ) is equal to 1 if the type of the  $j$ -th circular arc in  $T$  is  $R$  and it is equal to  $-1$  otherwise. Let  $\delta_{i,j}$  be equal to 1 if  $i = j$  and to 0 otherwise. Finally, if  $\vec{u}$  and  $\vec{v}$  are two vectors of  $\mathbb{R}^2$ ,  $\vec{u} \times \vec{v}$  denote their determinant (or, equivalently, the nonzero coordinates of their cross product, seen as vectors of  $\mathbb{R}^3$ ).

### First derivatives of $F_T(\theta_1, \theta_2)$

We start by a straightforward preliminary lemma.

**Lemma 32.** *If  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  are three vectors of  $\mathbb{R}^2$  such that  $\|\vec{v}\| = 1$ , and  $\vec{u} \cdot \vec{v} = 0$  or  $\vec{v} \cdot \vec{w} = 0$ , then*

$$\vec{u} \cdot \vec{w} = -(\vec{u} \times \vec{v})(\vec{v} \times \vec{w}).$$

*In particular, if  $\vec{u}$  and  $\vec{v}$  are two vectors such that  $\|\vec{v}\| = 1$ , then*

$$\vec{u} \cdot \frac{\partial \vec{v}}{\partial \theta_i} = -(\vec{u} \times \vec{v}) \left( \vec{v} \times \frac{\partial \vec{v}}{\partial \theta_i} \right)$$

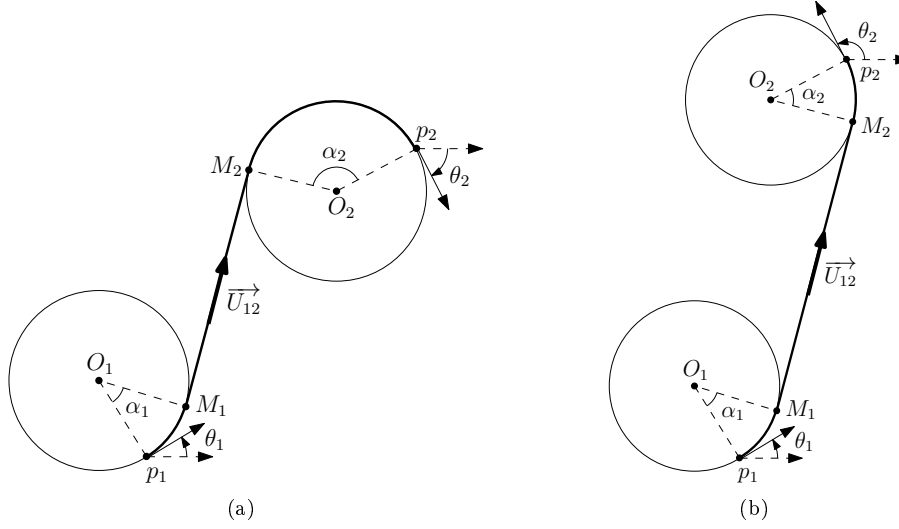
*Proof.* The proof of the first equality is straightforward:

$$\begin{aligned} \vec{u} \cdot \vec{w} &= \|\vec{u}\| \cdot \|\vec{w}\| \cos(\angle(\vec{u}, \vec{w})) \\ &= \|\vec{u}\| \cdot \|\vec{w}\| \cos(\angle(\vec{u}, \vec{v}) + \angle(\vec{v}, \vec{w})) \\ &= 0 - \|\vec{u}\| \cdot \|\vec{w}\| \sin(\angle(\vec{u}, \vec{v})) \sin(\angle(\vec{v}, \vec{w})) \\ &= -(\vec{u} \times \vec{v})(\vec{v} \times \vec{w}) \end{aligned}$$

The second equality of the lemma follows by considering  $\vec{w} = \frac{\partial \vec{v}}{\partial \theta_i}$  because, since  $\vec{v}$  has norm one, its scalar product with its derivative is zero.  $\square$

**Proposition 33.** *For a given path type  $T \in \{LSR, RSL, LSL, RSR\}$ , the length  $F_T(\theta_1, \theta_2)$  of the  $T$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is differentiable at any point  $(\theta_1, \theta_2)$  such that the corresponding  $T$ -path exists and both its circular arcs do not vanish. Furthermore,*

$$\frac{\partial F_T(\theta_1, \theta_2)}{\partial \theta_i} = \mu_{i=1} \mu_{C_i=R} (1 - \cos \alpha_i). \quad (5)$$

Figure 11: *LSR*- and *LSL*-paths from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$ .

*Proof.* Consider a  $T$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$ , with  $T \in \{LSR, RSL, LSL, RSR\}$  and refer to Figure 11. We determine, in turn, the derivatives with respect to  $\theta_i$  of the length  $\alpha_i$  of its  $i$ -th circular arc, and of the length  $g$  of its line segment. We then prove

$$\frac{\partial F_T(\theta_1, \theta_2)}{\partial \theta_i} = \mu_{i=1} \left( \mu_{C_i=R} + \overrightarrow{U_{12}} \times \overrightarrow{p_i O_i} \right) \quad (6)$$

which is equivalent to Eq. (5) because  $\overrightarrow{U_{12}} \times \overrightarrow{p_i O_i} = \mu_{C_i=L} \cos \alpha_i$  (see Figure 11(a)). Note that this yields that  $F_T$  is differentiable at any point  $(\theta_1, \theta_2)$  such that the corresponding  $T$ -path exists and none of its circular arcs vanishes, because, at such a point,  $F_T$  and its partial derivatives are defined and continuous.

If the length of the  $i$ -th ( $i = 1, 2$ ) circular arc is smaller than  $\pi$ , then  $\alpha_i = \arccos(\overrightarrow{O_i p_i} \cdot \overrightarrow{O_i M_i})$ , otherwise,  $\alpha_i = 2\pi - \arccos(\overrightarrow{O_i p_i} \cdot \overrightarrow{O_i M_i})$ . It follows that

$$\alpha_i(\theta_1, \theta_2) = \mu_{(\alpha_i < \pi)} \arccos(\overrightarrow{O_i p_i} \cdot \overrightarrow{O_i M_i}) \mod 2\pi$$

Its derivative with respect to  $\theta_j$ ,  $j = 1, 2$  is thus:

$$\frac{\partial \alpha_i}{\partial \theta_j} = \mu_{(\alpha_i < \pi)} \frac{-\left( \overrightarrow{O_i M_i} \cdot \frac{\partial \overrightarrow{O_i p_i}}{\partial \theta_j} + \overrightarrow{O_i p_i} \cdot \frac{\partial \overrightarrow{O_i M_i}}{\partial \theta_j} \right)}{\sqrt{1 - (\overrightarrow{O_i p_i} \cdot \overrightarrow{O_i M_i})^2}}$$

Since  $\overrightarrow{O_i p_i}$  and  $\overrightarrow{O_i M_i}$  have norm 1, the denominator can be simplified into  $|\overrightarrow{O_i p_i} \times \overrightarrow{O_i M_i}|$ . On the other hand, by Lemma 32, the terms of the numerator can be rewritten as follows:

$$\begin{aligned} \overrightarrow{O_i M_i} \cdot \frac{\partial \overrightarrow{O_i p_i}}{\partial \theta_j} &= -(\overrightarrow{O_i M_i} \times \overrightarrow{O_i p_i})(\overrightarrow{O_i p_i} \times \frac{\partial \overrightarrow{O_i p_i}}{\partial \theta_j}), \\ \overrightarrow{O_i p_i} \cdot \frac{\partial \overrightarrow{O_i M_i}}{\partial \theta_j} &= -(\overrightarrow{O_i p_i} \times \overrightarrow{O_i M_i})(\overrightarrow{O_i M_i} \times \frac{\partial \overrightarrow{O_i M_i}}{\partial \theta_j}). \end{aligned}$$

Furthermore,  $\overrightarrow{O_i p_i} = (\cos(\theta_i \pm \pi/2), \sin(\theta_i \pm \pi/2))$ , thus

$$\overrightarrow{O_i p_i} \times \frac{\partial \overrightarrow{O_i p_i}}{\partial \theta_j} = \delta_{i,j} \quad (7)$$

and

$$\overrightarrow{O_i M_i} \cdot \frac{\partial \overrightarrow{O_i p_i}}{\partial \theta_j} = \delta_{(i,j)} (\overrightarrow{O_i p_i} \times \overrightarrow{O_i M_i}).$$

Thus, if  $\varepsilon_i$  denotes the sign of  $\overrightarrow{O_i p_i} \times \overrightarrow{O_i M_i}$ , we have

$$\frac{\partial \alpha_i}{\partial \theta_j} = -\mu_{(\alpha_i < \pi)} \varepsilon_i \left( \delta_{(i,j)} - \overrightarrow{O_i M_i} \times \frac{\partial \overrightarrow{O_i M_i}}{\partial \theta_j} \right).$$

Now, one can easily observe (see Figure 11) that, for any  $i = 1, 2$ , we have  $\mu_{(\alpha_i < \pi)} \varepsilon_i = \mu_{i=1} \mu_{C_i=L}$ , and since also  $\overrightarrow{O_2 M_2} = \pm \overrightarrow{O_1 M_1}$ , we obtain

$$\frac{\partial \alpha_i}{\partial \theta_j} = -\mu_{i=1} \mu_{C_i=L} \left( \delta_{(i,j)} - \overrightarrow{O_j M_j} \times \frac{\partial \overrightarrow{O_j M_j}}{\partial \theta_j} \right).$$

Thus, if the path is of type *LSL* or *RSR*, then  $\mu_{i=1} \mu_{C_i=L}$  changes sign for  $i = 1, 2$ , and

$$\frac{\partial \alpha_1 + \alpha_2}{\partial \theta_j} = -\mu_{C_i=L} (\delta_{1,j} - \delta_{2,j}) = \mu_{C_j=R} \mu_{j=1}. \quad (8)$$

Otherwise, if the path is of type *LSR* or *RSL*, then  $\mu_{i=1} \mu_{C_i=L}$  is equal to  $\mu_{C_2=R}$ , and

$$\frac{\partial \alpha_1 + \alpha_2}{\partial \theta_j} = -\mu_{C_2=R} \left( 1 - 2 \overrightarrow{O_j M_j} \times \frac{\partial \overrightarrow{O_j M_j}}{\partial \theta_j} \right). \quad (9)$$

It remains to determine the derivative of  $g$ , the length of segment  $M_1 M_2$ . Since  $g^2 = \overrightarrow{M_1 M_2} \cdot \overrightarrow{M_1 M_2}$ , we have  $\frac{\partial g^2}{\partial \theta_j} = 2g \frac{\partial g}{\partial \theta_j} = 2 \overrightarrow{M_1 M_2} \cdot \frac{\partial \overrightarrow{M_1 M_2}}{\partial \theta_j}$ . Using the notation  $\overrightarrow{U_{12}} = \frac{\overrightarrow{M_1 M_2}}{M_1 M_2}$ , we thus have

$$\frac{\partial g}{\partial \theta_j} = \overrightarrow{U_{12}} \cdot \frac{\partial \overrightarrow{M_1 M_2}}{\partial \theta_j}.$$

We decompose  $\overrightarrow{M_1 M_2}$  into  $\overrightarrow{M_1 O_1} + \overrightarrow{O_1 p_1} + \overrightarrow{p_1 p_2} + \overrightarrow{p_2 O_2} + \overrightarrow{O_2 M_2}$ . Similarly as before, by Lemma 32,

$$\overrightarrow{U_{12}} \cdot \frac{\partial \overrightarrow{O_i p_i}}{\partial \theta_j} = - \left( \overrightarrow{U_{12}} \times \overrightarrow{O_i p_i} \right) \left( \overrightarrow{O_i p_i} \times \frac{\partial \overrightarrow{O_i p_i}}{\partial \theta_j} \right),$$

and, as noted above the last term is equal to  $\delta_{i,j}$  (Eq. 7) so

$$\overrightarrow{U_{12}} \cdot \frac{\partial \overrightarrow{O_i p_i}}{\partial \theta_j} = \delta_{i,j} (\overrightarrow{U_{12}} \times \overrightarrow{p_i O_i}). \quad (10)$$

Hence,

$$\begin{aligned} \overrightarrow{U_{12}} \cdot \frac{\partial \overrightarrow{O_1 p_1} - \overrightarrow{O_2 p_2}}{\partial \theta_j} &= \delta_{1,j} (\overrightarrow{U_{12}} \times \overrightarrow{p_1 O_1}) - \delta_{2,j} (\overrightarrow{U_{12}} \times \overrightarrow{p_2 O_2}) \\ &= \mu_{j=1} (\overrightarrow{U_{12}} \times \overrightarrow{p_j O_j}). \end{aligned}$$

Now, if the path is of type *LSL* or *RSR* then,  $\overrightarrow{M_1 O_1} + \overrightarrow{O_2 M_2} = 0$ , thus

$$\frac{\partial g}{\partial \theta_j} = \mu_{j=1} (\overrightarrow{U_{12}} \times \overrightarrow{p_j O_j}),$$

which, together with Eq. (8), concludes the lemma for paths of type *LSL* and *RSR*.

On the other hand, if the path is of type *LSR* or *RSL*, then  $\overrightarrow{M_1 O_1} = \overrightarrow{O_2 M_2}$ , and, by Lemma 32,

$$\overrightarrow{U_{12}} \cdot \frac{\partial \overrightarrow{O_2 M_2}}{\partial \theta_j} = - \left( \overrightarrow{U_{12}} \times \overrightarrow{O_2 M_2} \right) \left( \overrightarrow{O_2 M_2} \times \frac{\partial \overrightarrow{O_2 M_2}}{\partial \theta_j} \right).$$

Furthermore,  $\overrightarrow{U_{12}} \times \overrightarrow{O_2 M_2}$  is equal to  $\mu_{C_2=R}$  (see Figure 11(a)). Hence,

$$\frac{\partial g}{\partial \theta_j} = \mu_{j=1}(\overrightarrow{U_{12}} \times \overrightarrow{p_j O_j}) - 2 \mu_{C_2=R} \left( \overrightarrow{O_2 M_2} \times \frac{\partial \overrightarrow{O_2 M_2}}{\partial \theta_j} \right).$$

Finally, since  $\overrightarrow{O_2 M_2} \times \frac{\partial \overrightarrow{O_2 M_2}}{\partial \theta_j} = \overrightarrow{O_j M_j} \times \frac{\partial \overrightarrow{O_j M_j}}{\partial \theta_j}$ , Eq. (9) yields that

$$\frac{\partial F_T}{\partial \theta_j} = \mu_{j=1}(\overrightarrow{U_{12}} \times \overrightarrow{p_j O_j}) - \mu_{C_2=R}.$$

This concludes the proof of the lemma since, for paths of types  $LSR$  and  $RSL$ ,  $\mu_{C_2=R}$  is equal to  $-\mu_{j=1} \mu_{C_j=R}$ .  $\square$

## Second derivatives of $F_T(\theta_1, \theta_2)$

We start by computing the second derivatives of the length  $F_T(\theta_1, \theta_2)$  of a path of given type  $T \in \{LSR, RSL, LSL, RSR\}$ . We then compute its Hessian and show that it is positive definite, and thus that  $F_T(\theta_1, \theta_2)$  is locally convex, for any  $(\theta_1, \theta_2)$  such that both circular arcs of the corresponding path are shorter than  $\pi$ .

**Proposition 34.** *For a given path type  $T \in \{LSR, RSL, LSL, RSR\}$ , the length  $F_T(\theta_1, \theta_2)$  of the  $T$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is twice differentiable at any point  $(\theta_1, \theta_2)$  such that the corresponding  $T$ -path exists and none of its arcs vanishes. Furthermore,*

$$\frac{\partial^2 F_T}{\partial \theta_i \partial \theta_j} = \delta_{i,j} \sin \alpha_i + \frac{\sin \alpha_i \sin \alpha_j}{M_1 M_2} \quad (11)$$

and the determinant of the Hessian of  $F_T$  is

$$\sin \alpha_1 \sin \alpha_2 \left( 1 + \frac{\sin \alpha_1 + \sin \alpha_2}{M_1 M_2} \right).$$

*Proof.* We first prove

$$\frac{\partial^2 F_T}{\partial \theta_i \partial \theta_j} = \delta_{i,j} \mu_{i=1} \overrightarrow{U_{12}} \cdot \overrightarrow{p_i O_i} + \mu_{i=j} \frac{(\overrightarrow{U_{12}} \cdot \overrightarrow{p_i O_i})(\overrightarrow{U_{12}} \cdot \overrightarrow{p_j O_j})}{M_1 M_2} \quad (12)$$

which is equivalent to Eq. (11) because  $\mu_{i=1} \overrightarrow{U_{12}} \cdot \overrightarrow{p_i O_i} = \sin \alpha_i$  (see Figure 11). Note that, this will yield that,  $F_T$  is twice differentiable at any point  $(\theta_1, \theta_2)$  such that the corresponding  $T$ -path exists and none of its arcs vanishes, because at such a point  $F_T$  and all its first and second derivatives are defined and continuous.

By Proposition 33 (see Eq. (6)), we have

$$\frac{\partial F_T}{\partial \theta_i} = \mu_{i=1} \overrightarrow{U_{12}} \times \overrightarrow{p_i O_i} \pm 1,$$

and thus,

$$\frac{\partial^2 F_T}{\partial \theta_i \partial \theta_j} = \mu_{i=1} \left( \overrightarrow{U_{12}} \times \frac{\partial \overrightarrow{p_i O_i}}{\partial \theta_j} - \overrightarrow{p_i O_i} \times \frac{\partial \overrightarrow{U_{12}}}{\partial \theta_j} \right) \quad (13)$$

Notice, on the other hand that, given any three vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  of  $\mathbb{R}^2$  such that  $\|\vec{v}\| = 1$ ,

$$\vec{u} \times \vec{w} = (\vec{u} \times \vec{v})(\vec{v} \cdot \vec{w}) + (\vec{u} \cdot \vec{v})(\vec{v} \times \vec{w}). \quad (14)$$

Indeed,

$$\begin{aligned} \vec{u} \times \vec{w} &= \|\vec{u}\| \cdot \|\vec{w}\| \sin(\angle(\vec{u}, \vec{v})) \cos(\angle(\vec{v}, \vec{w})) + \|\vec{u}\| \cdot \|\vec{w}\| \cos(\angle(\vec{u}, \vec{v})) \sin(\angle(\vec{v}, \vec{w})) \\ &= (\vec{u} \times \vec{v})(\vec{v} \cdot \vec{w}) + (\vec{u} \cdot \vec{v})(\vec{v} \times \vec{w}) \end{aligned}$$

We can thus rewrite the terms of (13) using (14) as follows. Recall that since  $\overrightarrow{p_i O_i}$  has norm 1, its scalar product with its derivative is zero.

$$\overrightarrow{U_{12}} \times \frac{\partial \overrightarrow{p_i O_i}}{\partial \theta_j} = (\overrightarrow{U_{12}} \cdot \overrightarrow{p_i O_i}) \left( \overrightarrow{p_i O_i} \times \frac{\partial \overrightarrow{p_i O_i}}{\partial \theta_j} \right).$$

Furthermore, since  $\overrightarrow{p_i O_i} \times \frac{\partial \overrightarrow{p_i O_i}}{\partial \theta_j} = \delta_{i,j}$  (Eq. (7)), we get

$$\mu_{i=1} \overrightarrow{U_{12}} \times \frac{\partial \overrightarrow{p_i O_i}}{\partial \theta_j} = \mu_{i=1} \delta_{i,j} \overrightarrow{U_{12}} \cdot \overrightarrow{p_i O_i}, \quad (15)$$

which is the first term of (12).

Consider now the second term of (13). Again, using (14) and since  $\overrightarrow{U_{12}}$  has norm 1, its cross product with  $\overrightarrow{O_1 M_1}$  is zero, thus

$$\overrightarrow{p_i O_i} \times \frac{\partial \overrightarrow{U_{12}}}{\partial \theta_j} = (\overrightarrow{p_i O_i} \times \overrightarrow{O_1 M_1}) \left( \overrightarrow{O_1 M_1} \cdot \frac{\partial \overrightarrow{U_{12}}}{\partial \theta_j} \right).$$

Now, since  $\overrightarrow{U_{12}} = \overrightarrow{M_1 M_2} / M_1 M_2$ , its derivative is

$$\frac{\partial \overrightarrow{U_{12}}}{\partial \theta_j} = \frac{1}{M_1 M_2} \frac{\partial \overrightarrow{M_1 M_2}}{\partial \theta_j} - \frac{1}{M_1 M_2^2} \frac{\partial M_1 M_2}{\partial \theta_j} \overrightarrow{M_1 M_2}, \quad (16)$$

and, since  $\overrightarrow{O_1 M_1}$  and  $\overrightarrow{M_1 M_2}$  are orthogonal, we get:

$$\overrightarrow{O_1 M_1} \cdot \frac{\partial \overrightarrow{U_{12}}}{\partial \theta_j} = \frac{1}{M_1 M_2} \frac{\partial \overrightarrow{M_1 M_2}}{\partial \theta_j} \cdot \overrightarrow{O_1 M_1}.$$

We decompose  $\overrightarrow{M_1 M_2}$  into  $\overrightarrow{M_1 O_1} + \overrightarrow{O_1 p_1} + \overrightarrow{p_1 p_2} + \overrightarrow{p_2 O_2} + \overrightarrow{O_2 M_2}$ , and since the derivatives of  $\overrightarrow{M_1 O_1}$  and  $\overrightarrow{O_2 M_2}$  are orthogonal to  $\overrightarrow{O_1 M_1}$ , we obtain:

$$\begin{aligned} \overrightarrow{O_1 M_1} \cdot \frac{\partial \overrightarrow{U_{12}}}{\partial \theta_j} &= \frac{1}{M_1 M_2} \left( \frac{\partial \overrightarrow{O_1 p_1}}{\partial \theta_j} - \frac{\partial \overrightarrow{O_2 p_2}}{\partial \theta_j} \right) \cdot \overrightarrow{O_1 M_1} \\ &= \mu_{j=1} \frac{1}{M_1 M_2} \frac{\partial \overrightarrow{O_j p_j}}{\partial \theta_j} \cdot \overrightarrow{O_1 M_1} \end{aligned} \quad (17)$$

Furthermore, similarly as before,  $\overrightarrow{O_1 M_1} \cdot \frac{\partial \overrightarrow{O_j p_j}}{\partial \theta_j} = -\overrightarrow{O_1 M_1} \times \overrightarrow{O_j p_j}$ , by Lemma 32 and Eq. (7). Hence,

$$\overrightarrow{p_i O_i} \times \frac{\partial \overrightarrow{U_{12}}}{\partial \theta_j} = -\mu_{j=1} \frac{(\overrightarrow{p_i O_i} \times \overrightarrow{O_1 M_1})(\overrightarrow{O_j p_j} \times \overrightarrow{O_1 M_1})}{M_1 M_2}.$$

Finally, remark that, by Lemma 32,

$$\overrightarrow{U_{12}} \cdot \overrightarrow{p_i O_i} = -(\overrightarrow{U_{12}} \times \overrightarrow{O_1 M_1})(\overrightarrow{O_1 M_1} \times \overrightarrow{p_i O_i}),$$

and, since  $\overrightarrow{U_{12}} \times \overrightarrow{O_1 M_1} = \pm 1$ ,

$$\overrightarrow{p_i O_i} \times \frac{\partial \overrightarrow{U_{12}}}{\partial \theta_j} = -\mu_{j=1} \frac{(\overrightarrow{U_{12}} \cdot \overrightarrow{p_i O_i})(\overrightarrow{U_{12}} \cdot \overrightarrow{p_j O_j})}{M_1 M_2}.$$

Hence the second terms of (12) and (13) are equal, which concludes the proof of Eq. (12) (and thus of Eq. (11)) since, as we have seen in (15), the first terms of (11) and (13) are also equal.

This also concludes the proof of the proposition because the expression of the determinant of the Hessian follows directly from the expression of the second derivative (11):

$$\left( \sin \alpha_1 + \frac{\sin^2 \alpha_1}{M_1 M_2} \right) \left( \sin \alpha_2 + \frac{\sin^2 \alpha_2}{M_1 M_2} \right) - \left( \frac{\sin \alpha_1 \sin \alpha_2}{M_1 M_2} \right)^2 = \sin \alpha_1 \sin \alpha_2 \left( 1 + \frac{\sin \alpha_1 + \sin \alpha_2}{M_1 M_2} \right).$$

Note finally that one can easily prove that this determinant is also equal to  $\frac{-(\overrightarrow{U_{12}} \cdot \overrightarrow{p_1 O_1})(\overrightarrow{U_{12}} \cdot \overrightarrow{p_2 O_2})(\overrightarrow{U_{12}} \cdot \overrightarrow{p_1 p_2})}{M_1 M_2}$ .  $\square$

### Local convexity of $F_T(\theta_1, \theta_2)$

As we have already seen in Section 3, Proposition 34 yield the local convexity of the length function  $F_T$ . We recall it for ease of reading.

**Proposition 35.** *For a given path type  $T \in \{LSR, RSL, LSL, RSR\}$ , the length  $F_T(\theta_1, \theta_2)$  of the  $T$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is locally strictly convex at any point  $(\theta_1, \theta_2)$  such that the corresponding  $T$ -path exists, none of its arcs vanishes, and its two circular arcs have length less than  $\pi$ .*

*Proof.* Proposition 34 and the assumption that the length  $\alpha_i$  of each circular arc is in  $(0, \pi)$  imply that  $\frac{\partial^2 F_T}{\partial \theta_1^2}$  and the determinant of the Hessian of  $F_T$  are positive. Thus,  $F_T$  is positive definite, by Sylvester's criterion, and thus locally strictly convex at any point  $(\theta_1, \theta_2)$  that satisfies the hypotheses.  $\square$

### Geometric properties of $CSC$ -paths

We proved so far that, for a given path type  $T$  in  $\{LSR, RSL, LSL, RSR\}$ , the length  $F_T(\theta_1, \theta_2)$  of the  $T$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is locally convex at any  $(\theta_1, \theta_2)$  such that both circular arcs are shorter than  $\pi$ . We now prove that this property is also true for the length  $F(\theta_1, \theta_2)$  of a shortest  $CSC$ -path, that is for the function  $\min_{T \in \{LSR, RSL, LSL, RSR\}} F_T(\theta_1, \theta_2)$ . For that purpose, we prove the following geometric property of  $CSC$ -paths, which is interesting in its own rights.

**Proposition 36.** *If both circular arcs of a  $CSC$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  are strictly shorter than  $\pi$ , then all the other distinct  $CSC$ -paths are strictly longer.*

*Proof.* Consider two geometrically distinct paths of type  $T$  and  $T'$  in  $\{LSR, RSL, LSL, RSR\}$ , from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$ , such that both circular arcs of the  $T$ -path are strictly shorter than  $\pi$ . Similarly as for the  $T$ -path, let  $O'_i$  be the center of the unit circle supporting the  $i$ -th circular arc of the  $T'$ -path (defined by continuity if one of the circular arcs vanish), let  $\alpha'_i$  be its length, and let  $M'_1$  and  $M'_2$  be the first and last endpoint of the line segment of the  $T'$ -path.

In the first part of the proof, we show that the length of the line segment of the  $T$ -path is smaller than the one of the  $T'$ -path. In the second part, we show the same property about the circular arcs.

**Length of the line segments.** Note first that, the hypothesis that length of both circular arcs of the  $T$ -path are smaller than  $\pi$  implies that (see Figure 11(a)):

$$\begin{cases} \overrightarrow{p_1 O_1} \cdot \overrightarrow{M_1 M_2} \geq 0 \\ \overrightarrow{O_2 p_2} \cdot \overrightarrow{M_1 M_2} \geq 0 \end{cases} \quad (18)$$

By considering if necessary, the reverse paths possibly up to a symmetry, we can assume, without loss of generality, that the first circular arcs of the  $T$  and  $T'$ -paths have different orientations,  $R$  and  $L$ , respectively.

We prove that  $\|\overrightarrow{M_1 M_2}\| \leq \|\overrightarrow{M'_1 M'_2}\|$  by considering, in turn, the case where (i)  $T' = LSL$ , and otherwise, that is if  $T' = LSR$ , the cases where (ii)  $T = RSR$  or (iii)  $T = RSL$ .

**Case  $T' = LSL$ .** Refer, for instance, to Figure 12. In this case,  $\|\overrightarrow{M'_1 M'_2}\| = \|\overrightarrow{O'_1 O'_2}\|$  and  $\overrightarrow{O'_1 O'_2}$  can be decomposed as follow:

$$\overrightarrow{O'_1 O'_2} = \overrightarrow{O'_1 p_1} + \overrightarrow{p_1 O_1} + \overrightarrow{O_1 M_1} + \overrightarrow{M_1 M_2} + \overrightarrow{M_2 O_2} + \overrightarrow{O_2 p_2} + \overrightarrow{p_2 O'_2}.$$

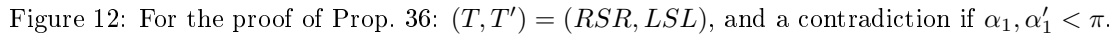
This implies that

$$\overrightarrow{O'_1 O'_2} \cdot \overrightarrow{M_1 M_2} \geq \overrightarrow{M_1 M_2} \cdot \overrightarrow{M_1 M_2};$$

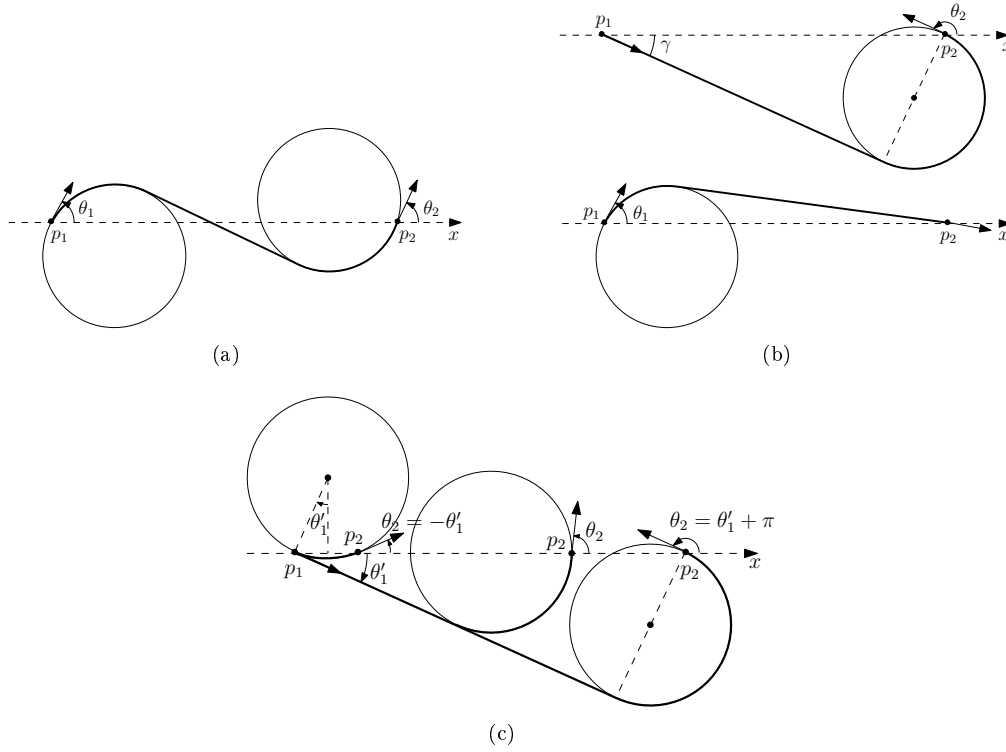
Indeed, first, depending on  $T$  and  $T'$ ,  $\overrightarrow{O'_1 p_1} + \overrightarrow{p_1 O_1}$  is equal either to  $\vec{0}$  or  $2\overrightarrow{p_1 O_1}$  and  $\overrightarrow{p_1 O_1} \cdot \overrightarrow{M_1 M_2} \geq 0$  by (18). Thus  $(\overrightarrow{O'_1 p_1} + \overrightarrow{p_1 O_1}) \cdot \overrightarrow{M_1 M_2} \geq 0$  and similarly for  $\overrightarrow{O_2 p_2} + \overrightarrow{p_2 O'_2}$ . This yields the above inequality since  $\overrightarrow{O_1 M_1}$  and  $\overrightarrow{M_2 O_2}$  are orthogonal to  $\overrightarrow{M_1 M_2}$ .

We thus get

$$\|\overrightarrow{O'_1 O'_2}\| \|\overrightarrow{M_1 M_2}\| \geq \overrightarrow{O'_1 O'_2} \cdot \overrightarrow{M_1 M_2} \geq \|\overrightarrow{M_1 M_2}\|^2$$



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Figure 13: For the proof of Prop. 36 in the case where  $T = RSL$ .

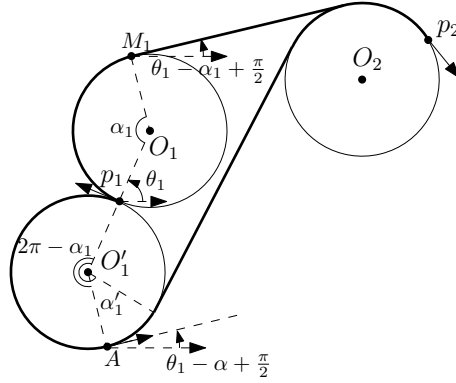
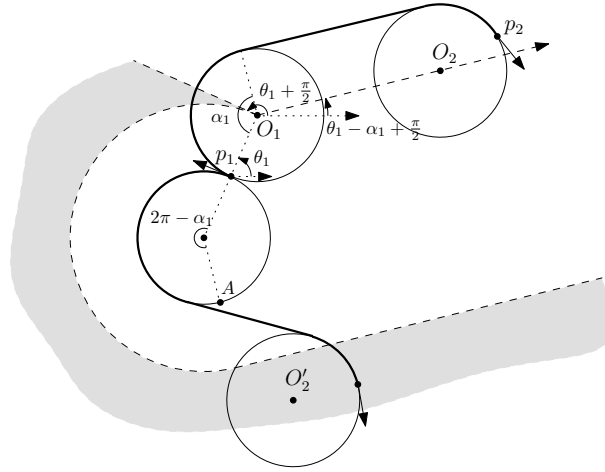
We now prove that (19) is satisfied which implies that  $\|\overrightarrow{M_1 M_2}\| \leq \|\overrightarrow{M'_1 M'_2}\|$ . Refer to Figure 13(a). The hypothesis that  $T = RSL$  implies that the two circular arcs lie on opposite sides of the line segment. Furthermore, since the second circular arc is shorter than  $\pi$ , if  $\theta_2$  decreases continuously with  $\theta_1$  fixed, both circular arcs shorten until one of them vanishes. Hence, there exists a path of type  $RS$  or  $SL$  from  $(p_1, \theta_1)$  to  $p_2$  whose circular arc is shorter than  $\pi$  (see Figure 13(b)). If the resulting modified path is of type  $RS$ , then the segment lies above the  $x$ -axis and, since the first circular arc is shorter than  $\pi$ ,  $\theta_1$  belongs to  $[0, \pi]$ . Similarly, if the modified path is of type  $SL$ , the segment lies below the  $x$ -axis and, since the second circular arc is shorter than  $\pi$ ,  $\theta_1$  belongs to  $[-\gamma, 0]$ , where  $\gamma = \arcsin \frac{2}{d}$  ( $d \geq 2$  since, otherwise, the circular arc is greater than  $\pi$ ). Similarly, if we fix  $\theta_2$  and continuously decrease  $\theta_1$ , both circular arcs shorten until the path is of type  $RS$  or  $SL$ , and we get, as before, that  $\theta_2$  belongs to  $[-\gamma, \pi]$ . Hence, both  $\theta_1$  and  $\theta_2$  are in  $[-\gamma, \pi]$ .

Now, if both  $\theta_1$  and  $\theta_2$  are in  $[0, \pi]$ , (19) is satisfied, and thus  $\|\overrightarrow{M_1 M_2}\| \leq \|\overrightarrow{M'_1 M'_2}\|$ . If  $\theta_1$  belong to  $[-\gamma, 0]$ , we decrease  $\theta_1$  with  $\theta_2$  fixed, until until one circular arc vanishes. Note that, while  $\theta_1$  decreases in  $[-\gamma, 0] \subseteq [-\pi/2, 0]$ ,  $\sin \theta_1 + \sin \theta_2$  decreases as well. The resulting modified path, from  $(p_1, \theta'_1)$  to  $(p_2, \theta_2)$ , is of type  $SL$  because, if it was of type  $RS$ , then  $\theta_1 \geq \theta'_1 > 0$ , contradicting the assumption that  $\theta_1 \in [-\gamma, 0]$  (if  $\theta'_1 = 0$ , the path is a segment and the path is also of type  $SL$ ). We now argue that  $\theta_2$  is in  $[-\theta'_1, \theta'_1 + \pi]$  and thus that  $\sin \theta_1 + \sin \theta_2 \geq \sin \theta'_1 + \sin \theta_2 \geq 0$ . Refer to Figure 13(c) and consider a path of type  $SL$  from  $(p_1, \theta'_1)$ , fixed, to a moving configuration  $(p_2, \theta_2)$  where  $d$  increases continuously from  $2 \sin |\theta'_1|$  to  $\frac{2}{\sin |\theta'_1|}$  (outside of this range, there is no path of type  $SL$  from  $(p_1, \theta'_1)$  to  $p_2$  with a circular arc shorter than  $\pi$ ). When  $d$  increases continuously from  $2 \sin |\theta'_1|$  to  $\frac{2}{\sin |\theta'_1|}$ ,  $\theta_2$  increases monotonically from  $-\theta'_1$  to  $\theta'_1 - \pi$ . Hence (19) is satisfied, and thus  $\|\overrightarrow{M_1 M_2}\| \leq \|\overrightarrow{M'_1 M'_2}\|$ .

Finally, if  $\theta_2$  belong to  $[-\gamma, 0]$ , we proceed similarly. We decrease  $\theta_2$  with  $\theta_1$  fixed until we get a path of type  $RS$  from  $(p_1, \theta_1)$  to  $(p_2, \theta'_2)$  (see Figure 13(b)). A symmetry about  $p_2$  yields a path of type  $SL$  from  $(p_2, \theta'_2)$  to  $(p'_1, \theta_1)$  and thus, as above we get that  $\theta_1$  is in  $[-\theta'_2, \theta'_2 + \pi]$  and thus that  $\|\overrightarrow{M_1 M_2}\| \leq \|\overrightarrow{M'_1 M'_2}\|$ .

This concludes the proof that  $\|\overrightarrow{M_1 M_2}\| \leq \|\overrightarrow{M'_1 M'_2}\|$  in all cases.



Figure 14: For the proof of Prop. 36, when  $\alpha_1 + \alpha'_1 > 2\pi$ .Figure 15: For the proof of Prop. 36, when  $T = RSR$  and  $T' = LSR$ .

**Length of the circular arcs.** We now show that the sum of the lengths of the circular arcs is strictly smaller on the  $T$ -path than on the  $T'$ -path. Recall that  $\alpha_i$  and  $\alpha'_i$  denote the length of the  $i$ -th circular arc of the  $T$ -path and  $T'$ -path, respectively.

Assume first that the first circular arcs of the  $T$  and  $T'$ -paths have different orientations ( $R$  or  $L$ ), and that the sum  $\alpha_1 + \alpha'_1$  of their lengths is larger than  $2\pi$ . Refer to Figure 14. Since the first circular arc of the  $T$ -path has length  $\alpha_1 < \pi$ , we can define the point  $A$  on the first circular arc of the  $T'$ -path such that its sub-path from  $p_1$  to  $A$  has length  $2\pi - \alpha_1$ . The sub-path from  $M_1$  to  $p_2$  on the  $T$ -path consists of a straight line segment and a circular arc which is shorter than  $\pi$ . Thus, the angular distance from  $M_1$  to  $p_2$  on the  $T$ -path is minimal, and the angular distance from  $A$  to  $p_2$  on the  $T'$ -path is necessarily larger or equal (since the polar angles of the oriented tangents to the two paths at  $M_1$  and  $A$  are equal). In other words, the total length of the circular arcs from  $A$  to  $p_2$  on the  $T'$ -path is larger than or equal to the length of the second circular arc of the  $T$ -path. On the other hand, the length of the circular arc from  $p_1$  to  $A$  on the  $T'$ -path is  $2\pi - \alpha_1$  which is greater than  $\alpha_1$  since  $\alpha_1 < \pi$ .

Hence, if the first circular arcs of the  $T$  and  $T'$ -paths have different orientations and  $\alpha_1 + \alpha'_1 > 2\pi$ , the total length of the circular arcs on the  $T$ -path is less than on the  $T'$ -path, i.e.,  $\alpha_1 + \alpha_2 < \alpha'_1 + \alpha'_2$ . We get the same result (by considering the reverse paths), if the second circular arcs of the  $T$  and  $T'$ -paths have different orientations and if  $\alpha_2 + \alpha'_2 > 2\pi$ .

Now, assume, without loss of generality, that the first circular arcs of the  $T$  and  $T'$ -paths have different orientations,  $R$  and  $L$ , respectively. Three cases occur: either the  $T$  and  $T'$  have the same orientation on the second circular arc, or  $(T, T')$  is equal to  $(RSL, LSR)$  or to  $(RSR, LSL)$ . We consider these three cases in turn, and show that  $\alpha_1 + \alpha_2 < \alpha'_1 + \alpha'_2$ .

**T and T' have the same orientation (R or L) on the second circular arc.** Assume first that  $T = RSR$  and  $T' = LSR$ , and refer to Figure 15.

We argue that the sum  $\alpha_1 + \alpha'_1$  of the lengths of the first circular arcs of the two paths is larger than  $2\pi$ , which, as shown above, yields the result that  $\alpha_1 + \alpha_2 < \alpha'_1 + \alpha'_2$ .

Since  $T = RSR$ ,  $O_2$  lies on the ray starting at  $O_1$  with polar angle  $\theta_1 - \alpha_1 + \frac{\pi}{2}$ . If the length  $\alpha'_1$  of the first circular arc of the  $LSR$ -path is smaller than or equal to  $2\pi - \alpha_1$ , then  $O'_2$  lies in the (closed) grey region of Figure 15. Since  $O_2 = O'_2$  lies in the intersection of this region and the ray, either  $\alpha_1 = 0$  or the line segment of the  $T$ -path vanishes. In both cases, the  $T$ -path is of type  $SR$  or  $R$  which are sub-types of  $LSR$ , and thus the  $T$  and  $T'$ -paths coincide, contradicting our hypothesis. Hence,  $\alpha'_1 > 2\pi - \alpha_1$ .

Note that we did not use here that the circular arcs of the  $T$ -path are shorter than  $\pi$ . Thus, if  $T = RSL$  and  $T' = LSL$ , we can exchange the roles of  $T$  and  $T'$  and we get, up to a symmetry, paths of type  $T = RSR$  and  $T' = LSR$ . Thus  $\alpha_1 + \alpha'_1 > 2\pi$  on the resulting (symmetric-exchanged) paths, and thus also on the initial path.

**Case (T, T') = (RSL, LSR).** If  $\alpha'_1 \geq \pi$  and  $\alpha'_2 \geq \pi$ , then, since  $\alpha_1 < \pi$  and  $\alpha_2 < \pi$ , we directly get the result that  $\alpha_1 + \alpha_2 < \alpha'_1 + \alpha'_2$ . We can thus assume that one of the circular arcs of the  $T'$ -path is shorter than  $\pi$ . Assume, without loss of generality, that this is its first arc, that is,  $\alpha'_1 < \pi$ .

We increase  $\theta_1$  continuously until  $\alpha_1$  or  $\alpha_2$  reaches  $\pi$ , or  $\alpha'_1$  or  $\alpha'_2$  reaches 0. Consider the  $T'$ -path: the fact that  $T' = LSR$  and that its first circular arc is shorter than  $\pi$  implies that, while  $\theta_1$  increases, both circular arcs shorten, that is  $\alpha'_1$  and  $\alpha'_2$  decrease (see Figure 13(a)). Similarly, for  $T = RSL$ , the fact that the first circular arc is shorter than  $\pi$  implies that, while  $\theta_1$  increases, both circular arcs get longer, that is  $\alpha_1$  and  $\alpha_2$  increase. Hence, while  $\theta_1$  increases,  $\alpha_1 + \alpha_2$  increases and  $\alpha'_1 + \alpha'_2$  decreases. It is thus sufficient to show that the resulting modified paths verify that  $\alpha_1 + \alpha_2 < \alpha'_1 + \alpha'_2$ .

If  $\alpha'_2$  reaches 0, the  $T'$ -path becomes of type  $LS$ ; this is a sub-type of  $LSL$  and we already proved, in the case where  $T = RSL$  and  $T' = LSL$ , that  $\alpha_1 + \alpha_2 < \alpha'_1 + \alpha'_2$ . The case where  $\alpha'_1$  reaches 0 is similar (by considering the reverse paths). On the other hand, if  $\alpha_1$  reaches  $\pi$ , then  $\alpha'_1 > \pi$ . Indeed, otherwise,  $O'_2$  lies in the intersection of the two grey regions of Figure 16, that is  $O'_2 = O_1$ , and the two paths are identical and of type  $R$ ; thus  $\alpha_2 = 0$  and, since  $\alpha_2$  increases during the motion, it is equal to 0 during the whole motion, and the initial  $T$ -path is of type  $RS$ ; similarly as above, this is a sub-type of  $RSR$ , and the case where  $T = RSR$  and  $T' = LSR$  was treated before. Thus, if  $\alpha_1$  reaches  $\pi$ , then  $\alpha'_1 > \pi$ . Hence,  $\alpha_1 + \alpha'_1 > 2\pi$ , which implies that  $\alpha_1 + \alpha_2 < \alpha'_1 + \alpha'_2$ . We get the same result if  $\alpha_2$  reaches  $\pi$ , by considering, for instance, the reverse path.

**Case (T, T') = (RSR, LSL).** Refer to Figure 12. Let  $\mathcal{E}$  (resp.  $\mathcal{E}'$ ) denote the (closed) half-plane delimited by the line tangent to the two paths at  $p_1$ , and containing the first circular arc of the  $T$ -path (resp.  $T'$ -path).

If  $\alpha_1$  and  $\alpha'_1$  are smaller than  $\pi$ , then the circles supporting the second circular arcs lie in  $\mathcal{E}$  and  $\mathcal{E}'$  respectively. Since these two circles have the point  $p_2$  in common, both paths are either identical and reduced to a segment or both paths have their two circular arcs of length  $\pi$ , contradicting the hypothesis in both cases. Hence,  $\alpha'_1 \geq \pi$  and  $\alpha_1 < \alpha'_1$ . Similarly,  $\alpha_2 < \alpha'_2$  and thus  $\alpha_1 + \alpha_2 < \alpha'_1 + \alpha'_2$ .

We thus have shown that if both circular arcs of a  $CSC$ -path are shorter than  $\pi$ , this path is strictly shorter than the other distinct  $CSC$ -paths, which concludes the proof of the proposition.  $\square$

## Local convexity of $F(\theta_1, \theta_2)$

We can now prove the local convexity of the length function  $F(\theta_1, \theta_2)$  of the shortest  $CSC$ -paths from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  on the domain of  $(\theta_1, \theta_2)$  such that both circular arcs are shorter than  $\pi$ . Figures 2(a) and 2(b) show an example of such domain and of the graph of  $F(\theta_1, \theta_2)$  over that domain.

**Theorem 37.** *The length  $F(\theta_1, \theta_2)$  of the shortest  $CSC$ -path from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$  is locally strictly convex at any point  $(\theta_1, \theta_2)$  such that both circular arcs of the corresponding path are strictly shorter than  $\pi$ .*

*Proof.* Theorem 37 essentially follows from Propositions 35 and 36, and from the fact that for any  $(\theta_1, \theta_2)$  such that two  $T$  and  $T'$ -paths coincide ( $T \neq T'$  in  $\{LSR, RSL, LSL, RSR\}$ ), that is when a circular arc vanishes, the first and second derivatives of  $F_T$  and  $F_{T'}$  are equal and thus  $F$  is locally  $C^2$  at this point.

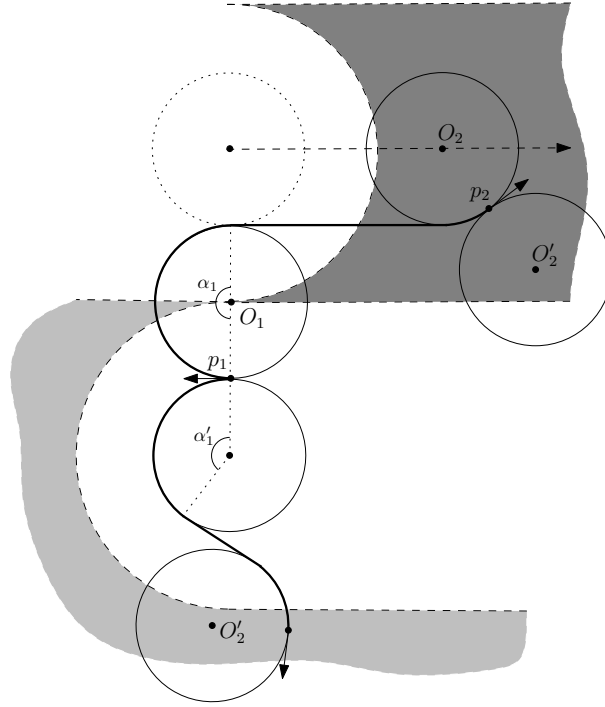


Figure 16: For the proof of Prop. 36: contradiction when  $T = RSL$ ,  $T' = LSR$ ,  $\alpha_1 = \pi$  and  $\alpha'_1 < \pi$ .

More precisely, consider a shortest *CSC*-path such that both its circular arcs are strictly shorter than  $\pi$ , and let  $T$  in  $\{LSR, RSL, LSL, RSR\}$  denote its type. (All the paths we consider here are from  $(p_1, \theta_1)$  to  $(p_2, \theta_2)$ .)

If both its circular arcs have nonzero length, it is geometrically distinct from all the other  $T'$ -paths, for  $T' \neq T$  in  $\{LSR, RSL, LSL, RSR\}$ . Furthermore, by Propositions 35 and 36, this  $T$ -path is strictly shorter than all other  $T'$ -paths, and its length function  $F_T$  is locally strictly convex at  $(\theta_1, \theta_2)$ . Hence,  $F = \min_{T \in \{LSR, RSL, LSL, RSR\}} F_T$  is also locally strictly convex at  $(\theta_1, \theta_2)$ .

Now, if one circular arc of the  $T$ -path has zero length, this path is geometrically equal to another path of type  $T' \neq T$ . We first show that  $F$  is locally strictly convex at such point by showing that  $F$  is locally  $C^2$ . However, note that this does not yet show the *strict* local convexity at such point. Recall that (see Propositions 33 and 34)

$$\begin{aligned} \frac{\partial F_T(\theta_1, \theta_2)}{\partial \theta_i} &= \mu_{i=1} \mu_{C_i=R} (1 - \cos \alpha_i) \\ \frac{\partial^2 F_T}{\partial \theta_i \partial \theta_j} &= \delta_{i,j} \sin \alpha_i + \frac{\sin \alpha_i \sin \alpha_j}{M_1 M_2}. \end{aligned} \quad (20)$$

Since the two paths are geometrically equal, the only values that differ in expressions of the partial derivatives of  $F_T$  and  $F'_{T'}$  is the  $\mu_{C_i=R}$  for which the  $C_i$  vanish(es). But then,  $\alpha_i = 0$  and thus  $\frac{\partial F_T(\theta_1, \theta_2)}{\partial \theta_i} = \frac{\partial F_{T'}(\theta_1, \theta_2)}{\partial \theta_i} = 0$ . Thus, the function  $F$  is locally  $C^2$  at such point, and thus locally convex.

We now show that the function  $F$  is locally strictly convex at any point  $\tilde{\Theta} = (\tilde{\theta}_1, \tilde{\theta}_2)$  such that two  $T$  and  $T'$ -paths are geometrically identical with a vanishing first circular arc and a second circular arc of length in  $(0, \pi)$ . We show that for  $\Theta = (\theta_1, \theta_2)$  small enough (in norm),  $F(\tilde{\Theta} + \Theta)$  is strictly above the tangent plane to the graph of  $F$  at  $(\tilde{\Theta}, F(\tilde{\Theta}))$ . Consider first the second-order Taylor expansion of  $F$  at  $\tilde{\Theta}$

$$F(\tilde{\Theta} + \Theta) = F(\tilde{\Theta}) + \Theta \cdot \nabla F + \frac{1}{2} \Theta^T H \Theta + o(\|\Theta\|^2)$$

where  $\nabla F$  and  $H$  denote the gradient and the Hessian of  $F$  at  $\tilde{\Theta}$ . Since  $F$  is  $C^2$  at  $\tilde{\Theta}$ , (20) yields that all the partial second derivatives of  $F$  at  $\tilde{\Theta}$  are zero except for  $\frac{\partial^2 F(\tilde{\Theta})}{\partial \theta_2^2}$  which is strictly positive. We thus

get that  $\Theta^T H \Theta = \frac{\partial^2 F(\tilde{\Theta})}{\partial \theta_2^2} \theta_2^2$  which is strictly positive unless  $\theta_2 = 0$ . When  $\theta_2 = 0$ , we consider the third-order Taylor expansion of  $F_T$  (the terms in  $\theta_1^2 \theta_2$ ,  $\theta_1 \theta_2^2$ , and  $\theta_2^3$  are zero since  $\theta_2 = 0$ ):

$$F_T(\tilde{\Theta} + \Theta) = F_T(\tilde{\Theta}) + \Theta \cdot \nabla F_T + \frac{1}{6} \frac{\partial^3 F_T(\tilde{\Theta})}{\partial \theta_1^3} \theta_1^3 + o(\|\Theta\|^3) \quad (21)$$

where  $\frac{\partial^3 F_T(\tilde{\Theta})}{\partial \theta_1^3}$  denotes the third derivative of  $F_T$  with respect to  $\theta_1$  at  $\tilde{\Theta}$ . Equation (20) gives that

$$\frac{\partial^3 F_T}{\partial \theta_1^3} = \frac{\partial \sin \alpha_1}{\partial \theta_1} \left( 1 + \frac{\sin \alpha_1}{M_1 M_2} \right) + \sin \alpha_1 \frac{\partial \left( 1 + \frac{\sin \alpha_1}{M_1 M_2} \right)}{\partial \theta_1}.$$

However, since the first circular arc vanishes, i.e.,  $\alpha_1 = 0$ , at  $\tilde{\Theta}$ , we get that

$$\frac{\partial^3 F_T(\tilde{\Theta})}{\partial \theta_1^3} = \frac{\partial \sin \alpha_1}{\partial \theta_1}.$$

Now,  $\sin \alpha_1 = \overrightarrow{U_{12}} \cdot \overrightarrow{p_1 O_1}$ , thus

$$\frac{\partial^3 F_T(\tilde{\Theta})}{\partial \theta_1^3} = \frac{\partial \overrightarrow{U_{12}}}{\partial \theta_1} \cdot \overrightarrow{p_1 O_1} + \overrightarrow{U_{12}} \cdot \frac{\partial \overrightarrow{p_1 O_1}}{\partial \theta_1}.$$

The second term is equal, by (10), to  $-\overrightarrow{U_{12}} \times \overrightarrow{p_1 O_1} = \mu_{C_1=R}$ . Thus, by (16), we get that

$$\frac{\partial^3 F_T(\tilde{\Theta})}{\partial \theta_1^3} = \left( \frac{1}{M_1 M_2} \frac{\partial \overrightarrow{M_1 M_2}}{\partial \theta_j} - \frac{1}{M_1 M_2^2} \frac{\partial M_1 M_2}{\partial \theta_j} \overrightarrow{M_1 M_2} \right) \cdot \overrightarrow{p_1 O_1} + \mu_{C_1=R}.$$

Furthermore, since  $\overrightarrow{M_1 M_2} \cdot \overrightarrow{p_1 O_1} = 0$  at  $\tilde{\Theta}$ ,

$$\frac{\partial^3 F_T(\tilde{\Theta})}{\partial \theta_1^3} = \frac{1}{M_1 M_2} \frac{\partial \overrightarrow{M_1 M_2}}{\partial \theta_j} \cdot \overrightarrow{p_1 O_1} + \mu_{C_1=R}.$$

Now, we decompose  $\overrightarrow{M_1 M_2}$  into  $\overrightarrow{M_1 O_1} + \overrightarrow{O_1 p_1} + \overrightarrow{p_1 O_2} + \overrightarrow{O_2 M_2}$ . The derivative of  $\overrightarrow{M_1 O_1} = \pm \overrightarrow{O_2 M_2}$  is orthogonal to  $\overrightarrow{M_1 O_1}$  which is equal to  $\overrightarrow{p_1 O_1}$  at  $\tilde{\Theta}$ , the derivative of  $\overrightarrow{O_1 p_1}$  is orthogonal to  $\overrightarrow{O_1 p_1}$ , and the derivative of  $\overrightarrow{p_1 O_2}$  with respect to  $\theta_1$  is zero. Hence,

$$\frac{\partial^3 F_T(\tilde{\Theta})}{\partial \theta_1^3} = \mu_{C_1=R}.$$

Finally, since the first circular arc vanishes in the shortest path with polar angle  $\tilde{\Theta}$ , we observe that the shortest path with polar angle  $\tilde{\Theta} + \Theta$ , with  $\theta_2 = 0$  and  $|\theta_1|$  small enough, is such that  $C_1$  is oriented  $R$  if  $\theta_1 > 0$ , and  $L$  if  $\theta_1 < 0$ . Thus  $\mu_{C_1=R} \theta_1^3 > 0$  for  $\theta_1 \neq 0$ . Therefore, we have proved that for  $\Theta$  small enough (in norm),  $F(\tilde{\Theta} + \Theta)$  is strictly above the tangent plane to the graph  $F$  at  $\tilde{\Theta}$ .

Hence, we have proved that  $F$  is locally strictly convex at any point such that one circular arc has length in  $[0, \pi)$  and the other has length in  $(0, \pi)$ . By continuity,  $F$  is also locally strictly convex at the point such that both circular arcs vanishes, which concludes the proof.  $\square$

## B Diamonds and lemons

Let  $\alpha \in (0, \pi]$  and recall that the lemon  $L_i^{i+1}(\alpha)$  is defined as the set of angles  $(\theta_i, \theta_{i+1})$  in  $(\mathbb{S}^1)^2$  such that both circular arcs of the shortest path from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  have length *strictly* less than  $\alpha$ . Also recall that the diamond  $D_i^{i+1}$  is defined as the image of the open quadrilateral with vertices  $(0, 2\pi)$ ,  $(\xi_i, \xi_i)$ ,  $(2\pi, 0)$ , and  $(2\pi - \xi_i, 2\pi - \xi_i)$  under the translation of vector  $(\nu_{i+1}^i, \nu_{i+1}^i)$ , where  $\xi_i = \frac{2\pi}{d_i - \frac{1}{d_i}}$  and  $d_i = |p_i p_{i+1}|$ . Here, we give a complete proof for Lemma 15, which states:

$$\text{If } |p_i p_{i+1}| \geq 4 \text{ then } L_i^{i+1}\left(\frac{3\pi}{4}\right) \subset D_i^{i+1} \subset L_i^{i+1}(\pi).$$

For the rest of the section, let  $d$  denote the distance  $d_i = |p_i p_{i+1}|$ .

## B.1 Translations and symmetries

We first identify certain symmetries of the regions  $L_i^{i+1}(\alpha)$  that will simplify the discussion.

**Translations of the lemon  $L_i^{i+1}(\alpha)$ .** First, we can assume that  $p_{i+1}$  lies at the origin, because translating  $\{p_i, p_{i+1}\}$  changes neither  $L_i^{i+1}(\alpha)$  nor  $D_i^{i+1}$ . Now, rotating  $p_i$  about  $p_{i+1}$  by any angle  $\mu$ , increases  $\theta_i$  and  $\theta_{i+1}$  by  $\mu$ , that is, translates  $L_i^{i+1}(\alpha)$  and  $D_i^{i+1}$  by vector  $(\mu, \mu)$ . This does not change their relative positions. We can thus assume that  $p_{i+1} = (0, 0)$ ,  $p_i = (d, 0)$ , and  $\nu_{i+1}^i = 0$ . Then, the lemon  $L_i^{i+1}(\alpha)$  and diamond  $D_i^{i+1}$  do not intersect the two hyperplanes  $\{(\theta_i, \theta_{i+1}) \in (\mathbb{S}^1)^2 \mid \theta_i = \nu_{i+1}^i = 0\}$  and  $\{(\theta_i, \theta_{i+1}) \in (\mathbb{S}^1)^2 \mid \theta_{i+1} = \nu_{i+1}^i = 0\}$  (as noted in the proofs of Lemmas 8 and 17). It follows that we can lift  $L_i^{i+1}(\alpha)$  and  $D_i^{i+1}$  from  $(\mathbb{S}^1)^2$  onto the square  $(0, 2\pi)^2$  of  $\mathbb{R}^2$ , and work in a Cartesian coordinate system  $(x, y)$ . Refer to Figure 7 with  $\nu_{i+1}^i = 0$  (or similarly to Figure 2(a)). Note that this lift is different from the lift  $\mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \Lambda_i \times \Lambda_{i+1}$  considered in Section 4, because, even though  $\Lambda_i = [0, 2\pi)$ , the endpoints of  $\Lambda_{i+1}$  are in  $\nu_{i+2}^{i+1} + 2\pi\mathbb{Z}$  which does not contain 0.

**Symmetries of  $L_i^{i+1}(\alpha)$ .** We first observe some symmetries of the region  $L_i^{i+1}(\alpha)$ . We define the following transformations on Dubins paths:

$s$  is the symmetry with respect to the  $x$ -axis. It maps the  $L_aSL_b$  (resp.  $L_aSR_b$ ,  $R_aSL_b$ ,  $R_aSR_b$ ) path from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  to the  $R_aSR_b$  (resp.  $R_aSL_b$ ,  $L_aSR_b$ ,  $L_aSL_b$ ) path from  $(p_i, 2\pi - \theta_i)$  to  $(p_{i+1}, 2\pi - \theta_{i+1})$ .

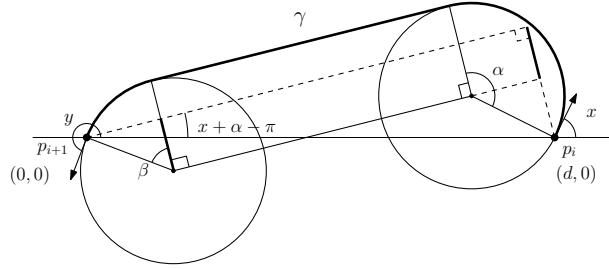
$r$  is composed of three transforms. We first reverse the path, turning the  $L_aSL_b$  (resp.  $L_aSR_b$ ,  $R_aSL_b$ ,  $R_aSR_b$ ) path from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  into the  $R_bSR_a$  (resp.  $L_bSR_a$ ,  $R_bSL_a$ ,  $L_bSL_a$ ) path from  $(p_{i+1}, \theta_{i+1} - \pi)$  to  $(p_i, \theta_i - \pi)$ . We then apply a symmetry with respect to the bisecting line of  $p_i$  and  $p_{i+1}$ , turning the  $R_bSR_a$  (resp.  $L_bSR_a$ ,  $R_bSL_a$ ,  $L_bSL_a$ ) path from  $(p_{i+1}, \pi + \theta_{i+1})$  to  $(p_i, \pi + \theta_i)$  into the  $L_bSL_a$  (resp.  $R_bSL_a$ ,  $L_bSR_a$ ,  $R_bSR_a$ ) path from  $(p_i, 2\pi - \theta_{i+1})$  to  $(p_{i+1}, 2\pi - \theta_i)$ . We finally apply the symmetry  $s$ , which turns the path into the  $R_bSR_a$  (resp.  $L_bSR_a$ ,  $R_bSL_a$ ,  $L_bSL_a$ ) path from  $(p_i, \theta_{i+1})$  to  $(p_{i+1}, \theta_i)$ .

The symmetry  $s$  implies that the shortest Dubins path from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  and the one from  $(p_i, 2\pi - \theta_i)$  to  $(p_{i+1}, 2\pi - \theta_{i+1})$  have equal length and that their longest circular arcs also have equal length. This implies that  $L_i^{i+1}(\alpha)$  is symmetric with respect to point  $(\pi, \pi)$ . Similarly, the transform  $r$  implies that the shortest Dubins path from  $(p_i, \theta_i)$  to  $(p_{i+1}, \theta_{i+1})$  and the one from  $(p_i, \theta_{i+1})$  to  $(p_{i+1}, \theta_i)$  have equal length and that their longest circular arcs also have equal length. Thus,  $L_i^{i+1}(\alpha)$  is symmetric with respect to the line  $y = x$ . By composing this symmetry and the symmetry about  $(\pi, \pi)$ , we get that  $L_i^{i+1}(\alpha)$  is also symmetric with respect to the line  $y = 2\pi - x$  (the line through  $(\pi, \pi)$  and orthogonal to  $y = x$ ).

## B.2 Boundary of $L_i^{i+1}(\alpha)$

We now give an analytical description of the boundary of  $L_i^{i+1}(\alpha)$ . As mentioned above, we consider  $L_i^{i+1}(\alpha)$  through the lift from  $(\mathbb{S}^1)^2$  to  $(0, 2\pi)^2$ . Its boundary thus lies in the closed square  $[0, 2\pi]^2$ . For simplicity, for any  $(x, y) \in [0, 2\pi]^2$ , the *path corresponding to  $(x, y)$*  refers to the shortest CSC-path from  $(p_i, x)$  to  $(p_{i+1}, y)$  whose longest circular arc has minimal length (in the case where several shortest CSC paths with minimal longest circular arc exist, we pick any of them).

We obtain our analytical description in three steps. We first show that any point on the boundary of  $L_i^{i+1}(\alpha)$  has the property that the longest circular arc of its corresponding path has length  $\alpha$ . We then give an analytical description of arcs of curves that are guaranteed to contain any point with that property. Let us emphasize that at this point, we will only have showed inclusions and do not claim that any point with the above property is on the boundary of  $L_i^{i+1}(\alpha)$  nor that any point on the arcs of curves have that property. Instead of trying to prove these reverse inclusions directly, in the third step, we argue that the union of the arcs forms a simple closed curve  $\sigma$  in  $[0, 2\pi]^2$ ; since  $L_i^{i+1}(\alpha)$  has nonempty interior and exterior regions, its boundary must disconnect  $\mathbb{S}^1 \times \mathbb{S}^1$  and therefore cannot be a proper subset of a simple closed curve.

Figure 17: For the expression of  $\beta$  when  $(x, y)$  is on  $\mathcal{C}_{L_\alpha SL \leq \alpha}$ .

**Step 1.** We first claim that the path corresponding to any point  $(x, y)$  on the boundary of  $L_i^{i+1}(\alpha)$  has its longest circular arc of length exactly  $\alpha$ . This statement may seem obvious, but it is not: one has to account for possible discontinuities in the function that maps  $(x, y)$  to the length of the longest circular arc of its corresponding path. If  $(x, y)$  is such a discontinuity then there must exist two shortest path from  $(p_i, x)$  to  $(p_{i+1}, y)$  with longest circular arcs of distinct lengths; this cannot occur in  $L_i^{i+1}(\pi)$  by Proposition 5. As a consequence we get that the path corresponding to any point  $(x, y)$  on the boundary of  $L_i^{i+1}(\alpha)$  has its longest circular arc of length either equal to  $\alpha$  or strictly larger than  $\pi$ . In the latter case, our definition of “corresponding path” implies that *any* shortest *CSC*-path from  $(p_i, x)$  to  $(p_{i+1}, y)$  has its longest circular arc of length strictly more than  $\pi$ ; it follows that in a neighborhood of  $(x, y)$ , the longest circular arc of the corresponding path remains strictly larger than  $\pi$ , and such a point  $(x, y)$  cannot be on the boundary of  $L_i^{i+1}(\alpha)$ . The claim follows.

**Step 2.** Let  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  and  $\mathcal{C}_{L_\alpha SR \leq \alpha}$  be the set of points  $(x, y) \in [0, 2\pi]^2$  whose corresponding path,  $\gamma$ , has type  $L_\alpha SL_\beta$  and  $L_\alpha SR_\beta$ , respectively, with  $\beta \leq \alpha$ . For technical reasons, it is convenient to consider  $(x, y)$  in the *open* square  $(0, 2\pi)^2$ , and to actually define  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  and  $\mathcal{C}_{L_\alpha SR \leq \alpha}$  by continuity over the closed square  $[0, 2\pi]^2$ .<sup>20</sup> We now describe analytically these curves.

Let us first consider  $(x, y) \in \mathcal{C}_{L_\alpha SL \leq \alpha}$ . Referring to Figure 17, we have:

$$y = x + \alpha + \beta. \quad (22)$$

Note that this equality is not modulo  $2\pi$ . Indeed, since  $\gamma$  is of type  $LSL$ ,  $\beta \leq \alpha \leq \pi$ , and  $p_{i+1}$  is left of  $p_i$  on the  $x$ -axis, we have that  $x$  is in  $(0, \pi]$  and  $y$  is in  $[\pi, 2\pi)$ . Since  $d \geq 4$ , the two circles supporting the circular arcs of  $\gamma$  are separated by a vertical line, which is crossed (from right to left) by the line segment of  $\gamma$ ; hence  $x + \alpha \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ . Furthermore, since  $\beta \leq \alpha$ , the oriented line segment of  $\gamma$  must point down, that is  $x + \alpha \in [\pi, \frac{3\pi}{2}]$ . Since  $\beta \leq \alpha \leq \pi$ , we thus have  $x + \alpha + \beta \in [\pi, \frac{5\pi}{2}]$ , and the fact that  $y \in [\pi, 2\pi)$  yields the claim.

Furthermore, since  $x + \alpha \in [\pi, \frac{3\pi}{2}]$ , we obtain the expression of  $\beta$  in terms of  $\alpha$ , by considering the length of the short bold segments in Figure 17:

$$\cos \beta = d \sin(x + \alpha - \pi) - \sin\left(\alpha - \frac{\pi}{2}\right) = \cos \alpha - d \sin(x + \alpha). \quad (23)$$

Plugging this expression in (22) we get that the curve  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  has the following equation; the upper bound on the domain of  $x$  corresponds to the condition  $\cos \alpha - d \sin(x + \alpha) \leq 1$  with  $\pi \leq x + \alpha \leq \frac{3\pi}{2}$ , and the lower bound comes from  $\pi \leq x + \alpha$ .

$$\begin{aligned} \mathcal{C}_{L_\alpha SL \leq \alpha} : \quad y &= x + \alpha + \arccos(\cos \alpha - d \sin(x + \alpha)) \\ \text{for } \pi - \alpha &\leq x \leq \pi - \alpha + \arcsin\left(\frac{1 - \cos \alpha}{d}\right). \end{aligned} \quad (24)$$

<sup>20</sup>When  $\alpha < \pi$  it is straightforward that  $\mathcal{C}_{L_\alpha SR \leq \alpha}$  and  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  are contained in  $(0, 2\pi)^2$ . When  $\alpha = \pi$ , there are exactly four points of the boundary of the square  $[0, 2\pi]^2$ , namely its corners, that have corresponding paths of type  $L_\pi SL_\beta$  or  $L_\pi SR_\beta$  with  $\beta \leq \pi$ . Although these four corners correspond to the same configurations when seen in  $(\mathbb{S}^1)^2$ ,  $(0, 2\pi)$  and  $(2\pi, 0)$  are the only ones that belong, as points of  $[0, 2\pi]^2$ , to the closure of the parts of  $\mathcal{C}_{L_\alpha SR \leq \alpha}$  and  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  contained in  $(0, 2\pi)^2$ ; we omit the other two points,  $(0, 0)$  and  $(2\pi, 2\pi)$ , since they are isolated and play no role in the boundary of  $L_i^{i+1}(\pi)$  in  $[0, 2\pi]^2$ .

Note that the lower bound on  $x$  corresponds to a  $LSL$  path whose circular arcs both have length  $\alpha$  (and whose segment has polar angle  $\pi$ ). The upper bound on  $x$  corresponds to  $\cos \beta = 1$ , that is, to a path whose second circular arc vanishes. A similar argument yields that the curve  $\mathcal{C}_{L_\alpha SR_{\leq \alpha}}$  has equation:

$$\begin{aligned} \mathcal{C}_{L_\alpha SR_{\leq \alpha}} : \quad & y = x + \alpha - \arccos(2 - \cos \alpha + d \sin(x + \alpha)) \\ \text{for } & \pi - \alpha + \arcsin\left(\frac{1 - \cos \alpha}{d}\right) \leq x \leq \pi - \alpha + \arcsin\left(\frac{2 - 2 \cos \alpha}{d}\right). \end{aligned} \quad (25)$$

The curves  $\mathcal{C}_{L_\alpha SL_{\leq \alpha}}$  and  $\mathcal{C}_{L_\alpha SR_{\leq \alpha}}$  meet in one of their endpoints (at  $x = \pi - \alpha + \arcsin(\frac{1 - \cos \alpha}{d})$  and  $y = x + \alpha$ ), which corresponds to a path of type  $L_\alpha S$ . Their union is thus a connected curve  $\tau$  whose endpoints are  $(\pi - \alpha, \pi + \alpha)$  on the line  $y = 2\pi - x$ , and  $(\tilde{x}, \tilde{x})$  on the line  $y = x$ , with  $\tilde{x} = \pi - \alpha + \arcsin(\frac{2 - 2 \cos \alpha}{d})$ . Moreover, we prove below (from (26) and (29)) that the slope of  $\tau$  is everywhere less than  $-1$ , which implies that  $\tau$  is simple and lies entirely in the quadrant  $x \leq y \leq 2\pi - x$ .

**Step 3.** Since any boundary point of  $L_i^{i+1}(\alpha)$  with corresponding path of type  $L_\alpha SL_{\leq \alpha}$  or  $L_\alpha SR_{\leq \alpha}$  must belong to  $\tau$ , the symmetries  $r$  and  $s$  yield that any boundary point of  $L_i^{i+1}(\alpha)$  must belong to  $\tau$  or one of its symmetric copies with respect to point  $(\pi, \pi)$ , to line  $y = x$ , and their combinations. We already know that  $\tau$  is simple and lies in the quadrant  $x \leq y \leq 2\pi - x$  with each endpoint on each of the two lines  $y = x$  and  $y = 2\pi - x$ . Thus, the union of  $\tau$  and its three symmetric copies forms a simple closed curve  $\sigma$  in  $[0, 2\pi]^2$ .

Let  $\hat{\sigma}$  denote the projection of  $\sigma$  on  $\mathbb{S}^1 \times \mathbb{S}^1$ . For  $\alpha < \pi$ ,  $\tau$  lies in the open square  $(0, 2\pi)^2$  and  $\hat{\sigma}$  is therefore a simple closed curve in  $\mathbb{S}^1 \times \mathbb{S}^1$ . Since both  $L_i^{i+1}(\alpha)$  and its complement have interior points, the boundary of  $L_i^{i+1}(\alpha)$  cannot be a proper subset of a simple closed curve and thus it must be equal to  $\hat{\sigma}$ . The situation for  $\alpha = \pi$  requires more care, as  $\tau$  lies in the open square  $(0, 2\pi)^2$  except for its endpoint with coordinates  $(0, 2\pi)$ . Thus,  $\sigma$  contains two points,  $(0, 2\pi)$  and  $(2\pi, 0)$ , which correspond to the same point in  $\mathbb{S}^1 \times \mathbb{S}^1$ . More precisely,  $\sigma$  consists of two curves that are simple, disjoint, and lie in  $(0, 2\pi)^2$  except for their endpoints which are equal to  $(0, 2\pi)$  and  $(2\pi, 0)$ . It follows that  $\hat{\sigma}$  is a closed curve in  $\mathbb{S}^1 \times \mathbb{S}^1$  consisting of two simple loops meeting in exactly one point:  $(0, 0)$ . We claim that each loop contains some point, other than  $(0, 0)$ , that belongs to the boundary of  $L_i^{i+1}(\pi)$ . Indeed, as shown previously, the junction between  $\mathcal{C}_{L_\pi SL_{\leq \pi}}$  and  $\mathcal{C}_{L_\pi SR_{\leq \pi}}$  is an interior point of  $\tau$  with corresponding path of type  $L_\pi S$ , and thus lies on the boundary of  $L_i^{i+1}(\pi)$ . By symmetry, the other (top) curve also contains a point on the boundary of  $L_i^{i+1}(\pi)$ . Together with the observation that both  $L_i^{i+1}(\pi)$  and its complement have interior points, this implies that the boundary of  $L_i^{i+1}(\pi)$  cannot be a proper subset of  $\hat{\sigma}$  but is equal to the whole curve.

We thus obtained a complete description of  $L_i^{i+1}(\alpha)$ . Note finally that the point  $(\pi, \pi)$ , which clearly belongs to any lemon, indicates on which side of this curve the lemon lies.

### B.3 Position of $D_i^{i+1}$

To complete the proof, it suffices to show that the segment bounding  $D_i^{i+1}$  in the quadrant  $x \leq y \leq 2\pi - x$  lies above the two curves  $\mathcal{C}_{L_\pi SL_{\leq \pi}}$  and  $\mathcal{C}_{L_\pi SR_{\leq \pi}}$ , and below the two curves  $\mathcal{C}_{L_{\frac{3\pi}{4}} SL_{\leq \frac{3\pi}{4}}}$  and  $\mathcal{C}_{L_{\frac{3\pi}{4}} SR_{\leq \frac{3\pi}{4}}}$  (as in Figure 7). Before completing the proof of the lemma, we first provide a very simple proof of a similar result in the case where  $d > 8.6$ .

#### B.3.1 Simple case where $d > 8.6$

Our simple proof uses a different diamond  $D_i^{i+1}$ , defined independently of  $d$ . Specifically, we let  $D_i^{i+1}$  be the image of the quadrilateral with vertices  $(0, 2\pi)$ ,  $(\frac{\pi}{4}, \frac{\pi}{4})$ ,  $(2\pi, 0)$ , and  $(\frac{7\pi}{4}, \frac{7\pi}{4})$  under the translation of vector  $(\nu_{i+1}^i, \nu_{i+1}^i)$ . The segment that bounds  $D_i^{i+1}$  in the quadrant  $x \leq y \leq 2\pi - x$  lies on the line  $y = 2\pi - 7x$  with  $x$  ranging from 0 to  $\frac{\pi}{4}$ . Consider the functions of  $x$  whose graphs are the curves  $\mathcal{C}_{L_\alpha SL_{\leq \alpha}}$  and  $\mathcal{C}_{L_\alpha SR_{\leq \alpha}}$ . A simple calculation shows that, for any  $\alpha$ , the derivative of these functions are less than  $-7$ .<sup>21</sup> We have seen in Step 2 of Section B.2 that the leftmost point of  $\mathcal{C}_{L_\pi SL_{\leq \pi}} \cup \mathcal{C}_{L_\pi SR_{\leq \pi}}$  is the point

<sup>21</sup>More precisely, the derivative of the function of  $\mathcal{C}_{L_\alpha SL_{\leq \alpha}}$  minus  $-7$  is  $8 + \frac{d \cos(x + \alpha)}{\sqrt{1 - (\cos(\alpha) - d \sin(x + \alpha))^2}}$ . Noting that the term in the square root is non-negative (by (23)) and that  $\cos(x + \alpha) \leq 0$  (since  $\pi \leq x + \alpha \leq \frac{3\pi}{2}$ ), the expression

$(0, 2\pi)$  which lies on the line  $y = 2\pi - 7x$ . It follows that this line is strictly above  $\mathcal{C}_{L_\pi SL \leq \pi} \cup \mathcal{C}_{L_\pi SR \leq \pi}$ , except for the endpoint  $(0, 2\pi)$ . On the other hand, a simple calculation also shows that the rightmost point of  $\mathcal{C}_{L_{\frac{3\pi}{4}} SL \leq \frac{3\pi}{4}} \cup \mathcal{C}_{L_{\frac{3\pi}{4}} SR \leq \frac{3\pi}{4}}$  is strictly above the line  $y = 2\pi - 7x$ , which implies that the line is strictly below this curve, and concludes the proof.

### B.3.2 Case where $d \geq 4$

Our proof consists of two parts considering the curves  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  and  $\mathcal{C}_{L_\alpha SR \leq \alpha}$ , independently. Let  $\ell$  be the line through points  $(0, 2\pi)$  and  $(\xi_i, \xi_i)$ , that is the line that contains the boundary segment of  $D_i^{i+1}$  lying in  $x \leq y \leq 2\pi - x$ . The idea of the proof is to bound the slope of  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  from above by the slope of  $\ell$  and to bound the values of the local extrema of  $\mathcal{C}_{L_\alpha SR \leq \alpha}$  using partial sums of its power series. In the following, we show how the proof is conducted but we do not detail all the calculations (which are done with Maple).

**Curves  $\mathcal{C}_{L_\alpha SL \leq \alpha}$ .** We show here that for any  $x$ , the slope of  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  is less than the slope of  $\ell$ . This will imply that  $\mathcal{C}_{L_\pi SL \leq \pi}$  is below  $\ell$  because the leftmost point of  $\mathcal{C}_{L_\pi SL \leq \pi}$  is  $(0, 2\pi)$ , which lies on  $\ell$ . This will also imply that  $\mathcal{C}_{L_{\frac{3\pi}{4}} SL \leq \frac{3\pi}{4}}$  lies above  $\ell$  once we proved that its rightmost point lies above  $\ell$ , which will come from the second part of the proof where we show that  $\mathcal{C}_{L_{\frac{3\pi}{4}} SR \leq \frac{3\pi}{4}}$  lies above  $\ell$ .

The slope of  $\ell$  is  $-(d - \frac{1}{d} - 1)$ , and that of  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  is

$$1 + \frac{d \cos(x + \alpha)}{\sqrt{1 - (\cos(\alpha) - d \sin(x + \alpha))^2}}. \quad (26)$$

Noting that  $\pi \leq x + \alpha \leq \frac{3\pi}{2}$ , for any  $x$ , the slope of  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  is less than the slope of  $\ell$  if and only if

$$d^2 \cos^2(x + \alpha) - \left(d - \frac{1}{d}\right)^2 \left(1 - (\cos(\alpha) - d \sin(x + \alpha))^2\right) > 0,$$

which is equivalent, after multiplying by  $d^2$  and replacing  $\sin(x + \alpha)$  by  $z$ , to

$$d^2 (d^4 - 3d^2 + 1) z^2 - 2 \cos(\alpha) d (d^2 - 1)^2 z + d^4 - (1 - \cos^2(\alpha)) (d^2 - 1)^2 > 0. \quad (27)$$

Consider this expression as a degree-two polynomial in  $z$ . The values of  $d$  for which its leading coefficient vanishes are all smaller than 2, and thus the leading coefficient is positive for any  $d \geq 2$ . On the other hand, the discriminant is a polynomial in  $d$ , which, after simplifying by  $4d^2$  and substituting  $d$  by  $u + 4$ , is

$$\begin{aligned} & (\cos^2 \alpha - 2)u^6 + (24 \cos^2 \alpha - 48)u^5 + (238 \cos^2 \alpha - 473)u^4 + (-2448 + 1248 \cos^2 \alpha)u^3 \\ & + (-7013 + 3649 \cos^2 \alpha)u^2 + (-10536 + 5640 \cos^2 \alpha)u + 3600 \cos^2 \alpha - 6479. \end{aligned}$$

Since all its coefficients are negative, it is negative for all  $u \geq 0$ , and thus the discriminant is negative for all  $d \geq 4$ . Therefore, (27) is satisfied, which completes the proof that the slope of  $\mathcal{C}_{L_\alpha SL \leq \alpha}$  is less than the slope of  $\ell$ , for any  $x$ .

**Curve  $\mathcal{C}_{L_\alpha SR \leq \alpha}$ .** Unlike in the proof for  $\mathcal{C}_{L_\alpha SL \leq \alpha}$ , it is not true that the slope of  $\mathcal{C}_{L_\alpha SR \leq \alpha}$  is less than that of  $\ell$ , for any  $x$  and  $d \geq 4$ . Thus, here we take a different approach, that is, to bound the values of local extrema.

Recall that we need to prove that (i)  $\mathcal{C}_{L_\pi SR \leq \pi}$  is below  $\ell$  and that (ii)  $\mathcal{C}_{L_{\frac{3\pi}{4}} SR \leq \frac{3\pi}{4}}$  is above  $\ell$ . Let  $F_\alpha$  be the function of  $x$  whose graph is  $\mathcal{C}_{L_\alpha SR \leq \alpha}$  minus the function whose graph is  $\ell$  (see (25)):

$$\begin{aligned} F_\alpha : \left[ \pi - \alpha + \arcsin\left(\frac{1 - \cos \alpha}{d}\right), \pi - \alpha + \arcsin\left(\frac{2 - 2 \cos \alpha}{d}\right) \right] &\rightarrow \mathbb{R} \\ x \mapsto x + \alpha - \arccos(2 - \cos \alpha + d \sin(x + \alpha)) - 2\pi + \left(d - \frac{1}{d} - 1\right) x. \end{aligned} \quad (28)$$

is negative if and only if  $64 \left(1 - (\cos(\alpha) + d \sin(x + \alpha))^2\right) - d^2 \cos^2(x + \alpha) < 0$ . This is a degree-two polynomial in  $\sin(x + \alpha)$  whose leading coefficient is  $-63d^2 < 0$  and discriminant is  $4d^2(-63d^2 + 4032 + 64 \cos^2 \alpha)$ , which is negative for any  $d \geq 8.6 > \sqrt{\frac{4032+64}{63}} \geq \sqrt{\frac{4032+64 \cos^2 \alpha}{63}}$ . Hence, for any  $x$  and  $d \geq 8.6$ , the slope of  $\mathcal{C}_{L_\alpha SR \leq \alpha}$  is less than that of  $y = 2\pi - 7x$ . The calculation is similar for  $\mathcal{C}_{L_\alpha SL \leq \alpha}$ .



Let  $x_{\min}$  and  $x_{\max}$  denote the leftmost and rightmost endpoints of the domain of definition of  $F_\alpha$ . We want to prove that  $F_\pi(x) < 0$  and  $F_{\frac{3\pi}{4}}(x) > 0$  for all  $x$  in  $[x_{\min}, x_{\max}]$ . For that purpose, we first prove that  $F_\alpha$  has at most two local extrema,  $x_-$  and  $x_+$ , other than  $x_{\min}$  and  $x_{\max}$ , and then that the above inequalities are satisfied for  $x_{\min}, x_{\max}, x_-$ , and  $x_+$ , for any  $d \geq 4$ .

**Local extrema of  $F$ .** We prove here that  $F_\alpha$  has at most two local extrema,  $x_\pm$ , other than its endpoints  $x_{\min}$  and  $x_{\max}$ . Since  $F_\alpha$  is continuous, we prove this by showing that (i) the denominator of  $F'_\alpha$  does not vanish on  $(x_{\min}, x_{\max})$ , and that (ii) the numerator of  $F'_\alpha$  admits two real roots (we do not need to argue if and when these roots belong to  $(x_{\min}, x_{\max})$ ).

The derivative of  $F_\alpha$  is

$$F'_\alpha(x) = \frac{d \cos(x + \alpha)}{\sqrt{1 - (-2 + \cos(\alpha) - d \sin(x + \alpha))^2}} + d - \frac{1}{d}. \quad (29)$$

The denominator vanishes when the square root term in  $F'_\alpha(x)$  vanishes, that is when  $-2 + \cos(\alpha) - d \sin(x + \alpha) = \varsigma = \pm 1$ . When  $\varsigma = -1$ , this is equivalent to  $\sin(x + \alpha) = -\frac{1 - \cos(\alpha)}{d}$ , and to  $x = x_{\min} = \pi - \alpha + \arcsin\left(\frac{1 - \cos(\alpha)}{d}\right)$  since  $\pi \leq x + \alpha \leq \frac{3\pi}{2}$ . When  $\varsigma = +1$ , the equality is equivalent to  $x = \pi - \alpha + \arcsin\left(\frac{3 - \cos(\alpha)}{d}\right)$ , which is larger or equal to  $x_{\max}$ ; indeed, this is equivalent to  $\arcsin\left(\frac{3 - \cos(\alpha)}{d}\right) \geq \arcsin\left(\frac{2 - 2\cos(\alpha)}{d}\right)$ , that is to  $3 - \cos(\alpha) \geq 2 - 2\cos(\alpha)$ , or  $\cos(\alpha) \geq -1$ . Hence, the denominator of  $F'_\alpha(x)$  does not vanish on  $(x_{\min}, x_{\max})$ .

The numerator of  $F'_\alpha$  is

$$d^2 \cos(x + \alpha) + (d^2 - 1) \sqrt{\delta} \quad \text{with} \quad \delta = 1 - (-2 + \cos(\alpha) - d \sin(x + \alpha))^2.$$

Since  $\cos(x + \alpha) \leq 0$  (by (25)) and  $d^2 - 1 > 0$  (for  $d \geq 4$ ), the numerator vanishes if and only if

$$\delta (d^2 - 1)^2 - d^4 \cos^2(x + \alpha) = 0. \quad (30)$$

This is a degree-two equation in  $\sin(x + \alpha)$ . Set  $z = \sin(x + \alpha)$  and denote by  $z_\pm$  its two roots. Then,

$$z_\pm = \frac{(d^2 - 1)^2 (2 - \cos(\alpha)) \pm \sqrt{\Delta}}{d (d^2 - (d^2 - 1)^2)}, \quad (31)$$

$$\text{with } \Delta = (d^2 - 1)^4 (2 - \cos(\alpha))^2 - (d^2 - (d^2 - 1)^2) \left( (d^2 - 1)^2 (1 - (2 - \cos(\alpha))^2) - d^4 \right).$$

Hence  $F_\alpha$  has at most two local extrema in the open interval  $(x_{\min}, x_{\max})$ , namely the values  $x_\pm = \pi - \arcsin(z_\pm) - \alpha$  (recall that  $x + \alpha \in [\pi, \frac{3\pi}{2}]$  by (25)).

**Sign of  $F_\alpha(x_\pm)$ .** Using  $\arccos u = \frac{\pi}{2} - \arcsin u$  and  $x = \pi - \arcsin(z) - \alpha$ , we can rewrite  $F_\alpha$  as

$$F_\alpha(x) = -\frac{5}{2} \pi + \alpha + \arcsin(2 - \cos(\alpha) + dz) + \left(d - \frac{1}{d}\right) (\pi - \alpha - \arcsin(z)). \quad (32)$$

For studying the sign of  $F_\alpha(x_\pm)$ , we bound  $\arcsin$  by partial sums of its power series (we need to use partial sums of degree at least 5 for the proof to work). We also need to determine the sign of the operands of  $\arcsin$ , that is, of  $z_\pm$  and  $2 - \cos(\alpha) + dz_\pm$ .

**Bounds on  $\arcsin$ .** Let  $\Gamma_1(z) = z + \frac{1}{6}z^3 + \frac{3}{40}z^5$  and  $\Gamma_2(z) = z + \frac{1}{6}z^3 + (\frac{\pi}{2} - \frac{7}{6})z^5$ ; note that  $\Gamma_1(z)$  is the partial sum, up to degree five, of the power series of  $\arcsin$ , and that the degree-five coefficient of  $\Gamma_2(z)$  is such that  $\Gamma_2(1) = \frac{\pi}{2} = \arcsin 1$ . We prove here that

$$\begin{aligned} \Gamma_1(z) &\leq \arcsin z \leq \Gamma_2(z) \quad \text{for } z \in [0, 1], \\ \Gamma_2(z) &\leq \arcsin z \leq \Gamma_1(z) \quad \text{for } z \in [-1, 0]. \end{aligned} \quad (33)$$

Note first that the second set of inequalities follows from the first set, since all three functions are odd. Also the inequality  $\Gamma_1(z) \leq \arcsin z$ , for  $z \geq 0$ , follows from the fact that all the terms of the power series of  $\arcsin$  are positive. We thus only have to prove that  $\arcsin z \leq \Gamma_2(z)$ , for  $z \in [0, 1]$ .

The derivative of  $\Gamma_2(z) - \arcsin z$  is  $1 + \frac{1}{2}z^2 + 5(\frac{\pi}{2} - \frac{7}{6})z^4 - \frac{1}{\sqrt{1-z^2}}$ , which is zero only if  $(1 + \frac{1}{2}z^2 + 5(\frac{\pi}{2} - \frac{7}{6})z^4)^2 (1 - z^2) - 1 = 0$ . The changes of variable  $z = 1/t$  and  $t = u+1$  transform the  $z$ -interval  $(0, 1)$  into the  $u$ -interval  $(0, +\infty)$ , and the polynomial into

$$(-447 + 180\pi)u^6 + (-2682 + 1080\pi)u^5 + (-6504 + 2610\pi)u^4 + (-8136 + 3240\pi)u^3 \\ + (-4064 + 1020\pi + 225\pi^2)u^2 + (-1560\pi + 992 + 450\pi^2)u - 36.$$

One can easily check that all the coefficients are positive except for the constant, thus by Descartes' rule of signs, the derivative of  $\Gamma_2(z) - \arcsin z$  vanishes exactly once in  $(0, +\infty)$ . Since  $\Gamma_2(z)$  and  $\arcsin z$  are equal for  $z = 0$  and  $z = 1$ , they cannot be equal for  $z \in (0, 1)$ . The result follows because the inequality is verified for  $z$  sufficiently small since  $\frac{\pi}{2} - \frac{7}{6} > \frac{3}{40}$  and all the terms of higher degree of the power series are negligible for  $z$  small enough.

**Signs of  $z_{\pm}$  and  $(2 - \cos \alpha + d z_{\pm})$ .** We prove here that  $z_{\pm} \leq 0$ ,  $(2 - \cos \alpha + d z_{+}) \leq 0$ , and  $(2 - \cos \alpha + d z_{-}) > 0$ . First note that  $z_{\pm} \leq 0$ , since  $z_{\pm} = \sin(x_{\pm} + \alpha)$  and  $\pi \leq x + \alpha \leq \frac{3\pi}{2}$  (by (25)). By (31), we have

$$-dz_{\pm} = \frac{(d^2 - 1)^2}{(d^2 - 1)^2 - d^2}(2 - \cos \alpha) \pm \frac{\sqrt{\Delta}}{(d^2 - 1)^2 - d^2}.$$

For  $d \geq 4$ , the coefficient of  $(2 - \cos \alpha)$  is larger than 1 and the denominator  $(d^2 - 1)^2 - d^2$  is positive, thus  $-dz_{+} \geq 2 - \cos \alpha$ . Concerning  $z_{-}$ , since the denominator of  $-dz_{-}$  is positive, we easily get that  $-dz_{-} < 2 - \cos \alpha$  if and only if  $d^2(2 - \cos \alpha) < \sqrt{\Delta}$ . Since both terms are positive, this is equivalent to

$$d^4(2 - \cos \alpha)^2 - \Delta < 0.$$

Making the change of variable  $d = u + 4$ , and setting  $G = (2 - \cos \alpha)^2 - 2$ , the left-hand side of the inequality is equal to

$$-Gu^6 - 24Gu^5 - (237G + 1)u^4 - (1232G + 16)u^3 - (3553G + 93)u^2 - (5384G + 232)u - 3344G - 209.$$

Since  $G > 0$  for  $\alpha \in [\frac{\pi}{2}, \pi]$ , the above expression is negative for all  $u \geq 0$ , which concludes the proof that  $2 - \cos \alpha + d z_{-} > 0$ .

**$F_{\pi}(\mathbf{x}_{\pm}) < 0$  and  $F_{\frac{3\pi}{4}}(\mathbf{x}_{\pm}) > 0$ .** Applying the bounds on  $\arcsin$  and the signs of  $z_{\pm}$ , we can now prove this claim. We first prove that  $F_{\pi}(x_{+}) < 0$ . Since  $2 - \cos \alpha + d z_{+} \leq 0$  and  $z_{+} \leq 0$ , (32) gives

$$F_{\pi}(x_{+}) = -\frac{5}{2}\pi + \pi + \arcsin(2 - \cos \pi + d z_{+}) + \left(d - \frac{1}{d}\right)(\pi - \pi - \arcsin z_{+}) \\ \leq -\frac{3}{2}\pi + \Gamma_1(3 + d z_{+}) - \left(d - \frac{1}{d}\right)\Gamma_2(z_{+}). \quad (34)$$

We replace  $z_{+}$  by its value which is of the form  $\frac{-b+\sqrt{\Delta}}{a}$ . As noted before, the denominator of  $z_{\pm}$  is negative for  $d \geq 4$ , and  $\Gamma_1$  and  $\Gamma_2$  are odd functions, thus the right hand side of (34) is negative if and only if its numerator is positive. This numerator is of the form  $A + B\sqrt{\Delta}$ . Substituting  $d = u + 4$  in  $A$  (resp.  $B$ ), we obtain a polynomial of degree 26 (resp. 20), whose coefficients are all positive (resp. negative). Thus  $A > 0$  and  $B < 0$  for  $d \geq 4$ , and  $A + B\sqrt{\Delta} > 0$  if  $A^2 - B^2\Delta > 0$ . We substitute, again,  $d = u + 4$  in  $A^2 - B^2\Delta$ , which gives a polynomial of degree 52 whose coefficients are all positive. Hence  $A^2 - B^2\Delta > 0$  for  $d \geq 4$ , which concludes the proof that  $F_{\pi}(x_{+}) < 0$ .

The proofs for the three other cases are similar. First, since  $2 - \cos \alpha + d z_{-} > 0$  and  $z_{-} \leq 0$ ,

$$F_{\pi}(x_{-}) \leq -\frac{3}{2}\pi + \Gamma_2(3 + d z_{-}) - \left(d - \frac{1}{d}\right)\Gamma_2(z_{-}).$$

Again, the right-hand side of the inequality is of the form  $A + B\sqrt{\Delta}$  over a negative denominator (for  $d \geq 4$ ), and we prove similarly as above that both  $A$  and  $B$  are positive for  $d \geq 4$ . Hence  $F_{\pi}(x_{-}) < 0$  for  $d \geq 4$ . Second,

$$F_{\frac{3\pi}{4}}(x_{-}) = -\frac{5}{2}\pi + \frac{3\pi}{4} + \arcsin\left(2 - \cos \frac{3\pi}{4} + d z_{-}\right) + \left(d - \frac{1}{d}\right)\left(\pi - \frac{3\pi}{4} - \arcsin z_{-}\right) \\ \geq -\frac{7}{4}\pi + \Gamma_1\left(2 + \frac{\sqrt{2}}{2} + d z_{-}\right) + \left(d - \frac{1}{d}\right)\left(\frac{\pi}{4} - \Gamma_1(z_{-})\right).$$

Again, the right-hand side is of the form  $A + B\sqrt{\Delta}$  over a negative denominator and, similarly as above,  $A < 0$  and  $B > 0$  for  $d \geq 4$ , and substituting  $d = u + 4$  in  $B^2\Delta - A^2$ , we get a polynomial of degree 54 whose coefficients are all negative. Hence  $F_{\frac{3\pi}{4}}(x_-) > 0$  for  $d \geq 4$ . Third,

$$F_{\frac{3\pi}{4}}(x_+) \geq -\frac{7}{4}\pi + \Gamma_2\left(2 + \frac{\sqrt{2}}{2} + dz_+\right) + \left(d - \frac{1}{d}\right)\left(\frac{\pi}{4} - \Gamma_1(z_+)\right).$$

Here the right-hand side of the inequality is of form  $A + B\sqrt{\Delta}$  over a negative denominator, and both  $A$  and  $B$  are negative for  $d \geq 4$ . Hence  $F_{\frac{3\pi}{4}}(x_+) > 0$  for  $d \geq 4$ .

**$F_\pi(\mathbf{x}) < 0$  and  $F_{\frac{3\pi}{4}}(\mathbf{x}) > 0$  for  $\mathbf{x} = \mathbf{x}_{\min}$  and  $\mathbf{x}_{\max}$ .** Recall that we already showed that  $\mathcal{C}_{L_\pi SL \leq \pi}$  is below  $\ell$  and that its rightmost point is equal to the leftmost point of  $\mathcal{C}_{L_\pi SR \leq \pi}$ . This implies that  $F_\pi(x_{\min}) < 0$ .

By (25),  $x_{\max} = \pi - \alpha + \arcsin\left(\frac{2-2\cos\alpha}{d}\right)$  and so,  $2 - \cos\alpha + d \sin(x_{\max} + \alpha) = \cos\alpha$ . Thus,

$$F_\alpha(x_{\max}) = -2\pi + \alpha - \arccos \cos\alpha + \left(d - \frac{1}{d}\right)x_{\max} = -2\pi + \left(d - \frac{1}{d}\right)x_{\max}.$$

For  $\alpha = \pi$ ,  $x_{\max} = \arcsin(4/d) \leq \Gamma_2(4/d)$ , which yields that  $F_\pi(x_{\max})$  is smaller or equal than a degree 6 polynomial in  $d$  over  $3d^6$ . The change of variable  $d = u + 4$  gives a polynomial whose coefficients are all negative which implies that  $F_\pi(x_{\max}) < 0$  for  $d \geq 4$ .

Similarly, for  $\alpha = \frac{3\pi}{4}$ ,  $x_{\max} = \frac{\pi}{4} + \arcsin\left(\frac{2+\sqrt{2}}{d}\right) \geq \frac{\pi}{4} + \Gamma_1\left(\frac{2+\sqrt{2}}{d}\right)$ . This yields that  $F_{\frac{3\pi}{4}}(x_{\max})$  is larger or equal to a degree 7 polynomial in  $d$  over  $120d^6$ . As before, the change of variable  $d = u + 4$  gives a polynomial whose coefficients are all positive; hence  $F_{\frac{3\pi}{4}}(x_{\max}) > 0$  for  $d \geq 4$ .

Finally, we prove that  $F_{\frac{3\pi}{4}}(x_{\min}) < 0$ . Recall that  $x_{\min} = \pi - \alpha + \arcsin\left(\frac{1-\cos\alpha}{d}\right)$ ; thus  $2 - \cos\alpha + d \sin(x_{\min} + \alpha) = 1$ , and

$$F_\alpha(x_{\min}) = -2\pi + \alpha + \left(d - \frac{1}{d}\right)x_{\min}.$$

For  $\alpha = \frac{3\pi}{4}$ ,  $x_{\min} = \frac{\pi}{4} + \arcsin\left(\frac{1+\sqrt{2}/2}{d}\right) \geq \frac{\pi}{4} + \Gamma_1\left(\frac{1+\sqrt{2}/2}{d}\right)$ . As before, this yields that  $F_{\frac{3\pi}{4}}(x_{\min})$  is larger or equal to a degree 7 polynomial in  $d$  over  $3840d^6$ , and the change of variable  $d = u + 4$  gives a polynomial whose coefficients are all positive. Hence  $F_{\frac{3\pi}{4}}(x_{\min}) > 0$  for  $d \geq 4$ , which concludes the proof of this lemma.

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<http://www.inria.fr>  
ISSN 0249-6399